REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
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Contents

1. CONVEX CONES AND GENERALIZATION OF KREIN MILMAN THEOREM

We want **to prove:** $\mathcal{P}_{3,4} = \sum_{3,4}$

To do it , we need several notions and intermediate results.

Definition 1.1. $C \subseteq \mathbb{R}^k$ is a **convex cone** if

$$\underline{x}, \underline{y} \in C \Rightarrow \underline{x} + \underline{y} \in C, \text{ and}$$
$$\underline{x} \in C, \lambda \in \mathbb{R}_+ \Rightarrow \lambda \underline{x} \in C$$

(i.e if it is closed under addition and under multiplication by non-negative scalars.)

Fact 1.2. $C \subseteq \mathbb{R}^k$ is a convex cone if and only if it is closed under non-negative linear combinations of its elements, i.e.
$\forall\ n \in \mathbb{N},\ \forall\ \underline{x}_1, \ldots, \underline{x}_n \in C, \forall\ \lambda_1, \ldots, \lambda_n \in \mathbb{R}_+ : \lambda_1 \underline{x}_1 + \ldots + \lambda_n \underline{x}_n \in C.$

Definition 1.3. Let $S \subseteq \mathbb{R}^k$. Then

$$\text{Cone}(S) := \{\text{non-negative linear combinations of elements from } S\}$$

is the convex cone generated by S.

Fact 1.4. For every $S \subseteq \mathbb{R}^k$, Cone(S) is the smallest convex cone which includes S.

Fact 1.5. If $S \subseteq \mathbb{R}^k$ is convex, then

$$\text{Cone}(S) := \{\lambda \underline{x} \mid \lambda \in \mathbb{R}_+, \underline{x} \in S\}.$$

Definition 1.6. $R \subseteq \mathbb{R}^k$ is a **ray** if $\exists\ \underline{x} \in \mathbb{R}^k,\ \underline{x} \neq 0$ s.t.

$$R = \{\lambda \underline{x} \mid \lambda \in \mathbb{R}_+\} := \underline{x}^+$$

(A ray R is a half-line.)

Definition 1.7. Let $C \subseteq \mathbb{R}^k$ be a convex set:

(1) a point $\underline{c} \in C$ is an **extreme point** if $C \setminus \{\underline{c}\}$ is convex.

(2) a ray $R \subseteq C$ is an **extreme ray** if $C \setminus R$ is convex.

Notation 1.8. Let $C \subseteq \mathbb{R}^k$ convex.

(1) ext(C) := set of all extreme points in C

(2) rext(C) := set of all extreme rays in C

Definition 1.9. (1) A **straight line** $L \subseteq \mathbb{R}^k$ is a translate of a 1-dimensional subspace, i.e. $L = \{\underline{x} + \lambda \underline{y} \mid \lambda \in \mathbb{R}\}$, for some $\underline{x}, \underline{y} \in \mathbb{R}^k, \underline{y} \neq 0$.

(2) $C \subseteq \mathbb{R}^k$ is **line free** if C contains no straight lines.

Theorem 1.10. (Klee) Let $C \subseteq \mathbb{R}^k$ be a closed line free convex set. Then

$$C = \text{cvx}\left(\text{ext}(C) \cup \text{rext}(C)\right)$$

Remark 1.11. (a) Let $C \subseteq \mathbb{R}^k$ be a convex cone and $\underline{x} \in C$, $\underline{x} \neq 0$. Then $\underline{x}$ is not extreme.
Also $\underline{x}^+ \subset C$.

(b) Let $C \subseteq \mathbb{R}^k$ be a line free convex cone. Then ext$(C) = \{0\}$.

Proof. If not, then $C \setminus \{0\}$ is not convex, so
$\exists\ \underline{x}, \underline{y} \in C \setminus \{0\}, \exists\ 0 < \lambda < 1$ s.t. $\lambda \underline{x} + (1-\lambda)\underline{y} \notin C \setminus \{0\}$.
But C is convex, so

$$\lambda \underline{x} + (1-\lambda)\underline{y} = \underline{0}.$$

That means that $\underline{x}^+ \cup \underline{y}^+$ is a straight line in C, a contradiction. □

Theorem 1.12.
Let $C \subseteq \mathbb{R}^k$ be a closed line free convex cone. Then
$$C = \text{cvx}\left(\text{rext}(C)\right)$$

Proof. By Remark 1.11, $\text{ext}(C) = \{0\}$.
Applying Theorem 1.10, we get $C = \text{cvx}\left(\text{rext}(C)\right)$. □

Remark 1.13. Let C be a line free convex cone

(1) $0 \neq \underline{x} \in C$ belongs to an extreme ray, i.e. $\underline{x}$ is **ray extreme** (equivalently, the ray $\{\lambda \underline{x} \mid \lambda \in \mathbb{R}_+\}$ generated by $\underline{x}$ is extreme) if and only if whenever $\underline{x} = \underline{x}_1 + \underline{x}_2$, with $\underline{x}_1, \underline{x}_2 \in C$, then $\underline{x}_i = \lambda_i \underline{x}$; $\lambda_i \in \mathbb{R}_+, \lambda_1 + \lambda_2 = 1$ (i.e. $\underline{x}_1, \underline{x}_2$ belong to the ray generated by $\underline{x}$).

(2) The set of convex linear combinations of points in extremal rays = the set of sums of points in extremal rays.

2. THE CONES $\mathcal{P}_{n,2d}$ and $\sum_{n,2d}$

Lemma 2.1. $\mathcal{P}_{n,2d}$ is a closed convex cone.

Proof. It is trivial that $\mathcal{P}_{n,2d}$ is a convex cone.
Next we prove that $\mathcal{P}_{n,2d}$ is closed:
Let $(P_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{P}_{n,2d}$ converging to P. Then for all $x \in \mathbb{R}^n, P_k(x) \to P(x)$.
We want (to show that) $P \in \mathcal{P}_{n,2d}$,
otherwise $\exists\ x_0 \in \mathbb{R}^n$, s.t. $P(x_0) = -\epsilon$, with $\epsilon > 0$.
And since $P_k(x_0) \to P(x_0)$ in $\mathbb{R}^n$, $\forall\ \epsilon > 0, \exists\ m \in \mathbb{N}$ s.t $\forall\ k > m : |P_k(x_0) - P(x_0)| < \epsilon$, thus (taking the same ϵ as above): $|P_k(x_0) + \epsilon| < \epsilon \Rightarrow P_k(x_0) < 0$, a contradiction (as $P_k \in \mathcal{P}_{n,2d}\ \forall\ k$). So $P \in \mathcal{P}_{n,2d}$, hence $\mathcal{P}_{n,2d}$ is closed. □

Lemma 2.2. The cone $\mathcal{P}_{n,2d}$ is line free.

Proof. Suppose not, then there exists a straight line L in $\mathcal{P}_{n,2d}$.
Write $L = \{F + \lambda G \mid \lambda \in \mathbb{R}\}$; $F, G \in \mathcal{P}_{n,2d}, G \neq 0$.
Since $-G \notin \mathcal{P}_{n,2d}$, take x_0 s.t. $-G(x_0) < 0$.

Then for (large enough λ i.e.) $\lambda \to -\infty$ we have $F(x_0) + \lambda G(x_0) < 0$
$\Rightarrow L \nsubseteq \mathcal{P}_{n,2d}$.
Hence $\mathcal{P}_{n,2d}$ is line free. □

Corollary 2.3. $\mathcal{P}_{n,2d}$ is the convex hull of its extremal rays.

Proof. By Lemma 2.1 and Lemma 2.2, $\mathcal{P}_{n,2d}$ is a line free closed convex cone. And therefore by the generalization of Krein-Milmann (Theorem 1.12) it is the convex hull of its extremal rays. □

Definition 2.4. A form $F \in \mathcal{P}_{n,2d}$ is **ray extremal** in $\mathcal{P}_{n,2d}$ if
$F = F_1 + F_2, F_1, F_2 \in \mathcal{P}_{n,2d} \Rightarrow F_i = \lambda_i F; i = 1, 2$ for $\lambda_i \in \mathbb{R}_+$ satisfying $\lambda_1 + \lambda_2 = 1$.

Similar definition for $\sum_{n,2d}$.

Note 2.5. By Remark 1.13 this just means that the ray generated by F is extremal.

Remark 2.6. (1) $F \in \sum_{n,2d}$ extremal $\Rightarrow F = G^2$ for some $G \in \mathcal{F}_{n,d}$.

(2) The converse of (1) is not true in general.
For example: $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ is not extremal in $\sum_{2,4}$.

(3) G^2 is extremal in $\sum_{n,2d} \not\Rightarrow G^2$ is extremal in $\mathcal{P}_{n,2d}$.
For instance Choi et al showed that
$p := f^2$, where $f(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 + (x^2y + y^2z - z^2x - xyz)^2$ is extremal in $\sum_{3,12}$ but not in $\mathcal{P}_{3,12}$.

Notation 2.7. We denote by $\mathcal{E}(\mathcal{P}_{n,2d})$ the set of all extremal forms in $\mathcal{P}_{n,2d}$.

Lemme 2.8. Let $E \in \mathcal{P}_{n,2d}$. Then $E \in \mathcal{E}(\mathcal{P}_{n,2d})$ if and only if $\forall\ F \in \mathcal{P}_{n,2d}$ with $E \geq F$ $\exists\ \alpha \in \mathbb{R}_+$ such that $F = \alpha E$.

Proof. ($\Rightarrow$) Let $E \in \mathcal{E}(\mathcal{P}_{n,2d}), F \in \mathcal{P}_{n,2d}$ s.t $E \geq F$, then
$G := E - F \in \mathcal{P}_{n,2d}$, so $E = F + G$.
Since E is extremal $\exists\ \alpha, \beta \geq 0, \alpha + \beta = 1$ such that $F = \alpha E$ and $G = \beta E$.

($\Leftarrow$) Let $F_1, F_2 \in \mathcal{P}_{n,2d}$ so that $E = F_1 + F_2$, then $E \geq F_1$, so $\exists\ \alpha \geq 0$ such that $F_1 = \alpha E$. Therefore $F_2 = E - F_1 = (1 - \alpha)E$ with $1 - \alpha \geq 0$ (since

$E, F_2 \in \mathcal{P}_{n,2d}$).
Thus E is extremal. □

Corollary 2.9. Every $F \in \mathcal{P}_{n,2d}$ is a finite sum of forms in $\mathcal{E}(\mathcal{P}_{n,2d})$.

Proof. By Corollary 2.3 and Remark 1.13 (2). □

3. PROOF OF $\mathcal{P}_{3,4} = \sum_{3,4}$

Corollary 2.9 is the first main item in the proof of Hilbert's Theorem (Theorem 2.8 of lecture 6) for the ternary quartic case. The second main item is the following lemma (which will be proved in the next lecture):

Lemma 3.1. Let $T(x, y, z) \in \mathcal{P}_{3,4}$. Then $\exists$ a quadratic form $q(x, y, z) \neq 0$ s.t. $T \geq q^2$, i.e. $T - q^2$ is psd.

Theorem 3.2. $\mathcal{P}_{3,4} = \sum_{3,4}$

Proof. Let $F \in \mathcal{P}_{3,4}$. By Corollary 2.9,
$F = E_1 + \ldots + E_k$, where E_i is extremal in $\mathcal{P}_{3,4}$ for $i = 1, \ldots, k$.
Applying Lemma 3.1 to each E_i we get
$E_i \geq q_i^2$, for some quadratic form $q_i \neq 0$
Since E_i is extremal, by Lemma 2.8, we get
$q_i^2 = \alpha_i E_i$; for some $\alpha_i > 0$, $\forall\ i = 1, \ldots, k$
and so $E_i = \left(\frac{1}{\sqrt{\alpha_i}} q_i\right)^2$ and hence $F \in \sum_{3,4}$. □