REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
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Contents

1. THE REAL SPECTRUM

Definition 1.1. Let A be a commutative ring with 1. We set:

$\boldsymbol{\mathcal{S}per}(A) := \{\alpha = (\mathfrak{p}, \leq) \mid \mathfrak{p}$ is a prime ideal of A and $\leq$ is an ordering on $ff(A/\mathfrak{p})\}$.

Note 1.2. $\mathcal{S}per(A) := \{\alpha = (\mathfrak{p}, \leq) \mid \mathfrak{p}$ is a real prime and $\leq$ an ordering on $ff(A/\mathfrak{p})\}$.

Definition 1.3. Let $\alpha = (\mathfrak{p}, \leq) \in \mathcal{S}per(A)$, then $\mathfrak{p} = \operatorname{Supp}(\alpha)$, the **Support** of α.

Recall 1.4. An **ordering** $P \subseteq A$ is a preordering with $P \cup -P = A$ and $\mathfrak{p} := P \cap -P$ prime ideal of A.

Definition 1.5. Alternatively, the **Real Spectrum** of A, $\mathcal{S}per(A)$ can be defined as:

$$\boldsymbol{\mathcal{S}per}(A) := \{P \mid P \subseteq A, P \text{ is an ordering of } A\}.$$

Remark 1.6. The two definitions of $\mathcal{S}per(A)$ are equivalent in the following sense:

The map

$$\varphi\colon \Big\{\text{Orderings in } A\Big\} \rightsquigarrow \Big\{(\mathfrak{p}, \leq), \mathfrak{p} \text{ real prime}, \leq \text{ordering on } ff\big(A/\mathfrak{p}\big)\Big\}$$

$$P \longmapsto \mathfrak{p} := P \cap -P, \leq_P \text{ on } ff\big(A/\mathfrak{p}\big)$$

$$\Big(\text{where } \frac{\overline{a}}{\overline{b}} \geq_P 0 \Leftrightarrow ab \in P \text{ with } \overline{a} = a + \mathfrak{p}\Big)$$

is bijective $\Big[$where $\varphi^{-1}\big(\mathfrak{p}, \leq\big)$ is $P := \big\{a \in A \mid \overline{a} \geq 0\big\}\Big]$. □

2. TOPOLOGIES ON $\mathcal{S}per(A)$

Definition 2.1. The **Spectral Topology** on $\mathcal{S}per(A)$:
$\mathcal{S}per(A)$ as a topological space, subbasis of open sets is:
$\mathcal{U}(a) := \big\{P \in \mathcal{S}per(A) \mid a \notin P\big\}$, $a \in A$.

(So a basis of open sets consists of finite intersection, i.e. of sets

$\mathcal{U}(a_1, \dots, a_n) := \big\{P \in \mathcal{S}per(A) \mid a_1, \dots, a_n \notin P\big\}$)

Then close by arbitrary unions to get all open sets.
This is called Spectral Topology.

Definition 2.2. The **Constructible (or Patch) Topology** on $\mathcal{S}per(A)$ is the topology that is generated by the open sets $\mathcal{U}(a)$ and their complements $\mathcal{S}per(A) \backslash \mathcal{U}(a)$, for $a \in A$.
(Subbasis for constructible topology is $\mathcal{U}(a)$, $\mathcal{S}per(A) \backslash \mathcal{U}(a)$, for $a \in A$.)

Remark 2.3. The constructible topology is finer than the Spectral Topology (i.e. more open sets).

Special case: $A = \mathbb{R}[\underline{X}]$

Proposition 2.4. There is a natural embedding

$$\mathcal{P} : \mathbb{R}^n \longrightarrow \mathcal{S}per\big(\mathbb{R}[\underline{X}]\big)$$

given by

$$\underline{x} \longmapsto P_{\underline{x}} := \Big\{f \in \mathbb{R}[\underline{X}] \mid f(\underline{x}) \geq 0\Big\}.$$

Proof. The map $\mathcal{P}$ is well defined.

Verify that $P_{\underline{x}}$ is indeed an ordering of A.

Clearly it is a preordering, $P_{\underline{x}} \cup -P_{\underline{x}} = \mathbb{R}[\underline{X}]$.

$\mathfrak{p} := P_{\underline{x}} \cap -P_{\underline{x}} = \Big\{ f \in \mathbb{R}[\underline{X}] \mid f(\underline{x}) = 0 \Big\}$ is actually a maximal ideal of $\mathbb{R}[\underline{X}]$, since $\mathfrak{p} = \text{Ker } (ev_{\underline{x}})$, the kernel of the evaluation map

$$ev_{\underline{x}} : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$$
$$f \longmapsto f(\underline{x})$$

so, $\frac{\mathbb{R}[\underline{X}]}{\mathfrak{p}} \simeq \underbrace{\mathbb{R}}_{\text{a field}}$ (by first isomorphism theorem)

$\Rightarrow \mathfrak{p}$ maximal $\Rightarrow \mathfrak{p}$ is prime ideal. □

Theorem 2.5. $\mathcal{P}(\mathbb{R}^n)$, the image of $\mathbb{R}^n$ in $\mathcal{S}per\big(\mathbb{R}[\underline{X}]\big)$ is dense in $\big(\mathcal{S}per\big(\mathbb{R}[\underline{X}]\big)$, Constructible Topology$\big)$ and hence in $\big(\mathcal{S}per\big(\mathbb{R}[\underline{X}]\big)$, Spectral Topology$\big)$. $\Big($i.e. $\overline{\mathcal{P}(\mathbb{R}^n)}^{patch} = \mathcal{S}per\big(\mathbb{R}[\underline{X}]\big)\Big)$.

Proof. By definition, a basic open set in $\mathcal{S}per\big(\mathbb{R}[\underline{X}]\big)$ has the form

$\mathcal{U} = \big\{ P \in \mathcal{S}per\big(\mathbb{R}[\underline{X}]\big) \mid f_i \notin P, g_j \in P; i = i, \ldots, s, j = 1, \ldots, t \big\}$, for some $f_i, g_j \in \mathbb{R}[\underline{X}]$.

Let $P \in \mathcal{U}$ $\Big($open neighbourhood of $P \in \mathcal{S}per\big(\mathbb{R}[\underline{X}]\big)\Big)$

We want to **show that:** $\exists\ \underline{y} \in \mathbb{R}^n$ s.t. $P_{\underline{y}} \in \mathcal{U}$

Consider $F = ff\big(\mathbb{R}[\underline{X}]/\mathfrak{p}\big); \mathfrak{p} = \text{Supp}(P) = P \cap -P$ and $\leq$ ordering on F induced by P.

Then $\big(F, \leq\big)$ is an ordered field extension of $\big(\mathbb{R}, \leq\big)$.

Consider $\underline{x} = \big(\overline{x_1}, \ldots, \overline{x_n}\big) \in F^n$, where $\overline{x_i} = X_i + \mathfrak{p}$

Then by definition of $\leq$ we have (as in the proof of PSS):

$f_i(\underline{x}) < 0$ and $g_j(\underline{x}) \geq 0$; $\forall\ i = i, \ldots, s,\ j = 1, \ldots, t$.

By Tarski Transfer, $\exists\ \underline{y} \in \mathbb{R}^n$ s.t.

$f_i(\underline{y}) < 0$ $\Big(\Leftrightarrow f_i \notin P_{\underline{y}} \Big)$ and $g_j(\underline{y}) \geq 0$ $\Big(\Leftrightarrow g_j \in P_{\underline{y}} \Big)$; $i = i, \ldots, s,\ j = 1, \ldots, t$

$\Leftrightarrow P_{\underline{y}} \in \mathcal{U}$ □

3. ABSTRACT POSITIVSTELLENSATZ

Recall 3.1. T proper preordering $\Rightarrow \exists\ P$ an ordering of A s.t. $P \supseteq T$.

Definiton 3.2. Let P be an ordering of A, fix $a \in A$. We define **Sign of** a **at** P :

$$a(P) := \begin{cases} 1 & \text{if } a \notin -P \\ 0 & \text{if } a \in P \cap -P \\ -1 & \text{if } a \notin P \end{cases}$$

(Note that this allows to consider $a \in A$ as a map on $\mathcal{S}per(A)$).

Notation 3.3. We write: $a > 0$ at P if $a(P) = 1$
$a = 0$ at P if $a(P) = 0$
$a < 0$ at P if $a(P) = -1$

Note that (in this notation) $a \geq 0$ at P iff $a \in P$.

Definition 3.4. Let $T \subseteq A$, then the **Relative Spectrum** of A with respect to T is

$$\mathcal{S}\boldsymbol{per}_T(A) = \{P \mid P \supseteq T; P \in \mathcal{S}per(A)\ \}.$$

Proposition 3.5. Let $T \subseteq A$ be a finitely generated preordering, say $T = T_S$; where $S = \{g_1, \dots, g_s\} \subseteq A$. Then

$$\begin{aligned}\mathcal{S}per_T(A) = \mathcal{S}per_S(A) &= \{P \in \mathcal{S}per(A) \mid g_i \in P\ ; i = i, \dots, s\} \\ &= \{P \in \mathcal{S}per(A) \mid g_i(P) \geq 0\ ; i = i, \dots, s\}\end{aligned}$$

□

Remark 3.5. Let $T \subseteq A$

(i) $\mathcal{S}per_T(A)$ inherits the relative spectral (respectively constructible) topology.

(ii) In case $T = T_{\{g_1,\dots,g_s\}}$ is a finitely generated preordering, then the proof of Theorem 2.5 goes through to give the following relative version for $\mathcal{S}per_T$:

Theorem 3.6. (Relative version of Theorem 2.5) Let $T = T_S =$ finitely generated preordering; $S = \{g_1, \dots, g_s\}$. Let $K = K_S = \{\underline{x} \in \mathbb{R}^n \mid g_i(\underline{x}) \geq 0\} \subseteq \mathbb{R}^n$, a basic closed semi-algebraic set. Consider ($\mathcal{S}per_T$, Constructible Topology). Then

$$\mathcal{P} : K \rightsquigarrow \mathcal{S}per_T\big(\mathbb{R}[\underline{X}]\big)$$

(defined as before)

$$\underline{x} \longmapsto P_{\underline{x}} = \Big\{ f \in \mathbb{R}[\underline{X}] \mid f(\underline{x}) \geq 0 \Big\}$$

is well defined $\big($i.e. $P_{\underline{x}} \supseteq T \ \forall \ \underline{x} \in K$ $\big)$.

Moreover $\mathcal{P}(K)$ is dense in $\Big(\mathcal{S}per_T\big(\mathbb{R}[\underline{X}]\big)$, Constructible Topology$\Big)$.

Proof. The proof is analogous to the proof of Theorem 2.5.
(Note the fact that T is finitely generated is crucial here to be able to apply Tarski Transfer.) □

Theorem 3.7. (Abstract Positivstellensatz) Let A be a commutative ring, $T \subseteq A$ be a preordering of A (not necessarily finitely generated). Then for $a \in A$:

(1) $a > 0$ on $\mathcal{S}per_T(A) \Leftrightarrow \exists \ p, q \in T$ s.t. $pa = 1 + q$

(2) $a \geq 0$ on $\mathcal{S}per_T(A) \Leftrightarrow \exists \ p, q \in T, m \geq 0$ s.t. $pa = a^{2m} + q$

(3) $a = 0$ on $\mathcal{S}per_T(A) \Leftrightarrow \exists \ m \geq 0$ s.t. $-a^{2m} \in T$.

Proof. (1) Let $a > 0$ on $\mathcal{S}per_T(A)$. Suppose for a contradiction that there are no elements $p, q \in T$ s.t. $pa = 1 + q$ i.e. s.t. $-1 = q - pa$
i.e. $-1 \neq q - pa \ \forall \ p, q \in T$
Thus $-1 \notin T' := T - Ta$.
$\Rightarrow T'$ is a proper preordering.
So (by recall 3.1) $\exists \ P$ an ordering of A with $T' \subseteq P$.
Now observe that $T \subseteq P$ i.e. $P \in \mathcal{S}per_T(A)$ but $-a \in P$ (i.e. $a(P) \leq 0$) i.e. $a \leq 0$ on P, a contradiction to the assumption. □

Proposition 3.8. Abstract Positivstellensatz $\Rightarrow$ Positivstellensatz.

Proof. $A = \mathbb{R}[\underline{X}], T = T_S = T_{\{g_1, \ldots, g_s\}}, K = K_S$.

It suffices **to show** (2) of PSS [Theorem 1.1 of lecture 03 on 20/04/10], i.e. $f \geq 0$ on $K_S \Leftrightarrow \exists \ m \in \mathbb{Z}_+, \exists \ p, q \in T_S$ s.t. $pf = f^{2m} + q$.

Let $f \in \mathbb{R}[\underline{X}]$ and $f \geq 0$ on K_S.

It suffices [by (2) of Theorem 3.7] to **show that** $f \geq 0$ on $\mathcal{S}per_T\big(\mathbb{R}[\underline{X}]\big)$:

If not then $\exists\ P \in \mathcal{S}per_T\big(\mathbb{R}[\underline{X}]\big)$ s.t. $f \notin P$
So, $P \in \mathcal{U}_T(f)$
$\Big($open neighbourhood of $P \in \mathcal{S}per_T\big(\mathbb{R}[\underline{X}]\big)\Big)$
Now by Theorem 3.6 (i.e. relative density of $\mathcal{P}(K)$ in $\mathcal{S}per_T\big(\mathbb{R}[\underline{X}]\big)$):
$\exists\ \underline{x} \in K$ s.t. $P_{\underline{x}} \in \mathcal{U}_T(f)$
$\Rightarrow f \notin P_{\underline{x}} \Rightarrow f(\underline{x}) < 0$, a contradiction to the assumption. $\square$