REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
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1. INTRODUCTION

Definiton 1.1. For $K \subseteq \mathbb{R}^n$,

Psd$(K) := \{f \in \mathbb{R}[\underline{X}] \mid f(\underline{x}) \geq 0 \ \forall \ \underline{x} \in K\}$.

Let $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[\underline{X}]$, then

$\mathbf{K_S} := \{\underline{x} \in \mathbb{R}^n \mid g_i(\underline{x}) \geq 0 \ \forall \ i = 1, \ldots, s\}$, the basic closed semi-algebraic set defined by S and

$\mathbf{T_S} := \Big\{ \sum_{e_1,\ldots,e_s \in \{0,1\}} \sigma_e \ g_1^{e_1} \ldots g_s^{e_s} \mid \sigma_e \in \Sigma\mathbb{R}[\underline{X}]^2, e = (e_1, \ldots, e_s)\Big\}$, the preordering generated by S.

We also introduce

$\mathbf{M_S} := \{\sigma_0 + \sigma_1 g_1 + \sigma_2 g_2 \ldots + \sigma_s g_s \mid \sigma_i \in \Sigma\mathbb{R}[\underline{X}]^2\}$, the quadratic module generated by S.

Remark 1.2. (i) M_S is a quadratic module in $\mathbb{R}[\underline{X}]$.

(ii) $M_S \subseteq T_S \subseteq \mathrm{Psd}(K_S)$.
(We shall study these inclusions in more detail later. In general these inclusions may be proper.)

(iii) $\mathrm{Psd}(K_S)$ is a preordering.

Definiton 1.3. T_S (resp. M_S) is called **saturated** if $\mathrm{Psd}(K_S) = T_S$ (resp. M_S).

2. EXAMPLES

For the examples that we are about to see, we need the following 2 lemmas:

Lemma 2.1. Let $f \in \mathbb{R}[\underline{X}]$; $f \not\equiv 0$, then $\exists\ \underline{x} \in \mathbb{R}^n$ s.t. $f(\underline{x}) \neq 0$. [Here n is such that $\underline{X} = (X_1, \dots, X_n)$.]

Proof. By induction on n.
If $n = 1$, result follows since a nonzero polynomial $\in \mathbb{R}[\underline{X}]$ has only finitely many zeroes.
Let $n \geq 2$ and $0 \not\equiv f \in \mathbb{R}[X_1, \dots, X_n] = \mathbb{R}[X_1, \dots, X_{n-1}][X_n]$.
$f \not\equiv 0 \Rightarrow f = g_0 + g_1 X_n + \dots + g_k X_n^k$; $g_0, g_1, \dots, g_k \in \mathbb{R}[X_1, \dots, X_{n-1}]$; $g_k \not\equiv 0$.
Since $g_k \not\equiv 0$, so by induction on n :
$\exists\ (x_1, x_2, \dots, x_{n-1})$ s.t. $g_k(x_1, x_2, \dots, x_{n-1}) \neq 0$.
$\Rightarrow$ The polynomial in one variable X_n i.e. $f(x_1, x_2, \dots, x_{n-1}, X_n) \not\equiv 0$.
Therefore by induction for $n = 1$, $\exists\ x_n \in \mathbb{R}$ s.t.
$f(x_1, x_2, \dots, x_{n-1}, x_n) \neq 0$ □

Remark 2.2. If $f \in \mathbb{R}[\underline{X}]$, $f \not\equiv 0$, then $\mathbb{R}^n \backslash Z(f) = \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$ is dense in $\mathbb{R}^n$, where $Z(f) := \{x \in \mathbb{R}^n \mid f(x) = 0\}$ is the zero set of f.
Equivalently, $Z(f)$ has empty interior. In other words, a polynomial which vanishes on a nonempty open set is identically the zero polynomial.

Lemma 2.3. Let $\sigma := f_1^2 + \dots + f_k^2$; $f_1, \dots, f_k \in \mathbb{R}[\underline{X}]$ and $f_1 \not\equiv 0$, then

(i) $\sigma \not\equiv 0$

(ii) $\deg(\sigma) = 2 \max\left\{\deg f_i \,; i = 1, \dots, k\right\}$
[In particular $\deg(\sigma)$ is even.]

Proof. (i) Since $f_1 \not\equiv 0$, so by lemma 2.1 $\exists\ \underline{x} \in \mathbb{R}^n$ s.t. $f_1(\underline{x}) \neq 0$.
$\Rightarrow \sigma(\underline{x}) = f_1(\underline{x})^2 + \dots + f_k(\underline{x})^2 > 0$
$\Rightarrow \sigma \not\equiv 0$.

(ii) $f_i = h_{i_0} + \dots + h_{i_d}$, where $d = \max\left\{\deg f_i \mid i = 1, \dots, k\right\}$; h_{i_j} homogeneous

of degree j or $h_{i_j} \equiv 0$ for $i = 1, \ldots, k$.
Clearly $\deg(\sigma) \leq 2d$.
To show $\deg(\sigma) = 2d$, consider the homogeneous polynomial
$h_{1_d}^2 + \ldots + h_{k_d}^2 := h_{2d}$
Note that if $h_{2d} \not\equiv 0$, then $\deg(h_{2d}) = 2d$ and h_{2d} is the homogeneous component of σ of highest degree (i.e. leading term), so $\deg(\sigma) = 2d$.
Now we know that $h_{i_d} \not\equiv 0$ for some $i \in \{1, \ldots, k\}$, so by (i) we get $h_{2d} \not\equiv 0$.
□

Now coming back to the inclusion: $T_S \subseteq \mathrm{Psd}(K_S)$

Example 2.4.(1) (i) $S = \phi$, $n = 1 \Rightarrow K_S = \mathbb{R}$ and $T_S = \sum \mathbb{R}[X]^2$
$\Rightarrow T_S = \mathrm{Psd}(\mathbb{R})$.

(ii) $S = \left\{(1 - X^2)^3\right\}$, $n = 1 \Rightarrow K_S = [-1, 1]$ (compact),
$T_S = \left\{\sigma_0 + \sigma_1(1 - X^2)^3 \mid \sigma_0, \sigma_1 \in \sum \mathbb{R}[X]^2\right\} = M_S$.
Claim. $T_S \subsetneq \mathrm{Psd}(K_S)$
For example: $(1 - X^2) \in \mathrm{Psd}[-1, 1]$ (clearly),
but $(1 - X^2) \notin T_S$, since if we assume for a contradiction that

$$(1 - X^2) = \sigma_0 + \sigma_1(1 - X^2)^3, \tag{1}$$

where $\sigma_0 \not\equiv 0$, $\sigma_0 = \sum f_i^2$, then evaluating (1) at $x = \pm 1$ we get
$\sigma_0(\pm 1) = \sum f_i^2(\pm 1) = 0$
$\Rightarrow f_i(\pm 1) = 0$
$\Rightarrow f_i = (1 - X^2)g_i$, for some $g_i \in \mathbb{R}[X]$
$\Rightarrow \sigma_0 = (1 - X^2)^2 \sum g_i^2$
Substituting σ_0 back in (1) we get

$$1 = (1 - X^2) \sum g_i^2 + (1 - X^2)^2 \sigma_1 \tag{2}$$

Evaluating (2) at $x = \pm 1$ yields $1 = 0$, a contradiction.

(iii) $S = \left\{X^3\right\}, n = 1 \Rightarrow K_S = [0, \infty)$ (noncompact),
$T_S = \left\{\sigma_0 + \sigma_1 X^3 \mid \sigma_0, \sigma_1 \in \sum \mathbb{R}[X]^2\right\} = M_S$.
Claim. $T_S \subsetneq \mathrm{Psd}(K_S)$
For example: $X \in \mathrm{Psd}(K_S)$, but $X \notin T_S$ (we will use degree argument to show this).
We compute the possible degrees of elements $t \in T_S$; $t \not\equiv 0$
Let

$$t = \sigma_0 + \sigma_1 X^3;\ \sigma_0, \sigma_1 \in \sum \mathbb{R}[X]^2,$$

then

- $\sigma_0 \not\equiv 0 \Rightarrow \deg(\sigma_0)$ is even.
- $\sigma_1 \not\equiv 0 \Rightarrow \deg(\sigma_1)$ is even.
- $\sigma_0 \equiv 0 \Rightarrow \deg(t)$ is odd and ≥ 3.
- $\sigma_1 \equiv 0 \Rightarrow \deg(t)$ is even.
- $\sigma_0 \not\equiv 0,\ \sigma_1 \not\equiv 0$, then

  [even =] $\deg(\sigma_0) \neq \deg(\sigma_1)$ [= odd]

  So, $\deg(t) = \max\left\{\deg(\sigma_0), \deg(\sigma_1)\right\}$ is even or odd ≥ 3.

This proves that $X \notin T_S$ and hence $T_S \subsetneq \mathrm{Psd}(K_S)$. □

Example 2.4.(2) $S = \phi$, $n = 2 \Rightarrow K_S = \mathbb{R}^2$ and $T_S = M_S = \sum \mathbb{R}[X, Y]^2$.
We see that $T_S \subsetneq \mathrm{Psd}(K_S)$
For example: $m(X, Y) := X^2Y^4 + X^4Y^2 - 3X^2Y^2 + 1 \in \mathrm{Psd}(\mathbb{R}^2)$, but $\notin T_S = \sum \mathbb{R}[X, Y]^2$.

3. POSITIVSTELLENSATZ (Geometric Version)

Theorem 3.1. (Positivstellensatz: Geometric Version) Let $A = \mathbb{R}[\underline{X}]$. Let $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[\underline{X}]$, K_S, T_S as defined above, $f \in \mathbb{R}[\underline{X}]$. Then

(1) $f > 0$ on $K_S \Leftrightarrow \exists\, p, q \in T_S$ s.t. $pf = 1 + q$

(2) $f \geq 0$ on $K_S \Leftrightarrow \exists\, m \in \mathbb{N}_0, \exists\, p, q \in T_S$ s.t. $pf = f^{2m} + q$

(3) $f = 0$ on $K_S \Leftrightarrow \exists\, m \in \mathbb{N}_0$ s.t. $-f^{2m} \in T_S$

(4) $K_S = \phi \Leftrightarrow -1 \in T_S$.

Important **corollaries** to the PSS are:

(i) The real Nullstellensatz

(ii) Hilbert's 17^{th} problem

(iii) Abstract Positivstellensatz

The proof of the PSS consists of two parts:
-Step I: prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1)

-Step II: prove (4) [using Tarski Transfer]

We shall start the proof with step II:

Clearly $K_S \neq \phi \Rightarrow -1 \notin T_S$ (since $-1 \in T_S \Rightarrow K_S = \phi$), so it only remains to prove the following proposition:

Proposition 3.2. If $-1 \notin T_S$ (i.e. if T_S is a proper preordering), then $K_S \neq \phi$.

For proving this we need to recall some definitions and results:

Definition 3.3.1. Let A be a commutative ring with 1, a preordering $P \subseteq A$ is said to be an **ordering** on A if $P \cup -P = A$ and $\mathfrak{p} := P \cap -P$ is a prime (hence proper) ideal of A.

Definition 3.3.2. Let P be an ordering in A, then Support$P := \mathfrak{p}$ (the prime ideal $P \cap -P$).

Lemma 3.4.1. Let A be a commutative ring with 1. Let P be a maximal proper preordering in A. Then P is an ordering.

Lemma 3.4.2. Let A be a commutative ring with 1 and $P \subseteq A$ an ordering. Then P induces uniquely an ordering on $F := ff(A/\mathfrak{p})$ defined by:

$\forall\, a, b \in A, \dfrac{\overline{a}}{\overline{b}} \geq_P 0$ (in F) $\Leftrightarrow ab \in P$, where $\overline{a} = a + \mathfrak{p}$.