REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
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Contents

1. THE POLYNOMIAL RING $\mathbb{R}[\underline{X}]$

Notation 1.1. $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$ is the polynomial ring in n variables and real coefficients, where $\mathbb{R}$ is the set of real numbers.

Note that $\mathbb{R}[\underline{X}]$ is a vector space of countable dimension (a basis is $\{\underline{X}^{\underline{\alpha}} \mid \underline{\alpha} \in \mathbb{Z}^n_+\}$, where $\underline{X}^{\underline{\alpha}} := X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ is a monomial).

Definition 1.2. A polynomial is said to be **homogenous** if it is a linear combination of monomials with same degree (or zero polynomial).

Convention: deg (0) := $-\infty$, where ˝0˝ is the polynomial with 0 coefficients.

Definition 1.3. Let $f \in \mathbb{R}[\underline{x}]$, the **homogenous decomposition** of f is $f = h_0 + \ldots + h_d$, where h_i are homogenous (or 0) and $\deg(h_i) = i$ if $h_i \not\equiv 0$.

Note that if $h_d \not\equiv 0$, then $d = \deg(h_d) = \deg(f)$.

Remark 1.4. Let $f, g \in \mathbb{R}[\underline{x}]$; $f \not\equiv 0$, $g \not\equiv 0$, then:

(i) $\deg(fg) = \deg(f) + \deg(g)$

(ii) $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$

(iii) $\deg(f + g) = \max\{\deg(f), \deg(g)\}$, if $\deg(f) \neq \deg(g)$.

2. BOREL MEASURE

Definition 2.1. Let X be a locally compact Hausdorff topological space (ie. $\forall\, x \in X\ \exists\, \mathcal{U} \ni x$ such that $\overline{\mathcal{U}}$ is compact). A **Borel measure** "μ" on X is a positive measure such that every $B \in \beta^{\delta}(X)$ is measurable, where $\beta^{\delta}(X) :=$ the smallest class of subsets of X which contain all compact sets and is closed under finite unions, complements and countable intersections.

Further we will assume that μ is **regular**, ie.
$\forall\, B \in \beta^{\delta}(X), \forall\, \epsilon > 0\ \ \exists\, C, \mathcal{U} \in \beta^{\delta}(X)$ with $C \subseteq B \subseteq \mathcal{U}$, where C is compact, $\mathcal{U}$ is open and $\mu(C) + \epsilon \geq \mu(B) \geq \mu(\mathcal{U}) - \epsilon$.

Definition 2.2. Let K be a closed compact subset of $\mathbb{R}^n$. K is said to be **basic closed semi-algebraic** if there exists a finite $S \subseteq \mathbb{R}[\underline{X}]$, say $S = \{g_1, \ldots, g_s\}$ (for $s \in \mathbb{N}$) such that $K = K_S = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0\ \forall\, i = 1, \ldots, s\}$.

Notation 2.3. $\sum \mathbb{R}[\underline{X}]^2 := \{\sigma = \sum_{i=1}^{m} f_i^2 \mid f_i \in \mathbb{R}[\underline{X}], m \in \mathbb{N}\}$.

Theorem 2.4. (Schmüdgen's Positivstellensatz) Let $K \subseteq \mathbb{R}^n$ be a compact semi-algebraic set, $K = K_S$ (as above). Let $L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$ be a linear functional. Then L can be represented by a positive Borel measure μ defined on K $\Big($ie. $L(f) = \int_K f d\mu$ for $f \in \mathbb{R}[\underline{X}]\Big)$ if and only if $L(\sigma g_1^{e_1} \ldots g_s^{e_s}) \geq 0\ \forall\, \sigma \in \sum \mathbb{R}[\underline{X}]^2$ and $e_1, \ldots, e_s \in \{0, 1\}$.

See Corollary 2.6 in lecture 13.

3. PREORDERING

Definition 3.1. Let A be a commutative ring with 1,
$\Sigma A^2 := \{\Sigma a_i^2 \mid i \geq 0, a_i \in A\}$.

(1) A **quadratic module** M in A is a subset $M \subseteq A$ such that
$M + M \subseteq M$, $a^2 M \subseteq M\ \forall\, a \in A, 1 \in M$.

(2) A **preordering** T in A is a quadratic module with $TT \subseteq T$.
T is said to be **proper** if $-1 \notin T$.

Remark 3.2. If $\frac{1}{2} \in A$ then $T = A$ is the only preordering in A that is not proper.

Proof. For $a \in A$ one can write: $a = (\frac{a+1}{2})^2 + (-1)(\frac{a-1}{2})^2 \in T$ □

Examples 3.3.

(1) $\underbrace{\Sigma A^2}_{\text{(the smallest preordering)}} \subseteq T$ for a preordering T in A.

(2) Let $S = \{g_1, \ldots, g_s\} \subseteq A$, then

$$T_S := \Big\{ \sum_{e_1,\ldots,e_s \in \{0,1\}} \sigma_e \, g_1^{e_1} \ldots g_s^{e_s} \mid \sigma_e \in \Sigma A^2, e = (e_1, \ldots, e_s) \Big\}$$

is the preordering generated by $g_1, \ldots, g_s$.

Definiton 3.4. A preordering $T \subseteq A$ is said to be **finitely generated** if $\exists$ a finite $S \subseteq A$ with $T = T_S$.

For example: ΣA^2 is finitely generated with $S = \phi$.

Example 3.5. Let $S \subseteq A = \mathbb{R}[\underline{X}]$ be a finite subset. We associate to S the basic closed semi-algebraic subset $K_S \subseteq \mathbb{R}^n$ and the finitely generated preordering $T_S \subseteq \mathbb{R}[\underline{X}]$. We recall that $K_S := \{\underline{x} \in \mathbb{R}^n \mid g_i(\underline{x}) \geq 0 \ \forall \ i = 1, \ldots, s\}$, $S = \{g_1, \ldots, g_s\}$.

For example: If $S = \phi : K_S = \mathbb{R}^n$, $T_S = \sum \mathbb{R}[\underline{X}]^2$.

Definiton 3.6. An element $f \in T_S$ is said to be **positive semidefinite** on K_S if $f(\underline{x}) \geq 0$ for all $\underline{x} \in K_S$.

For $K \subseteq \mathbb{R}^n$, set $\mathrm{Psd}(K) := \Big\{ f \in \mathbb{R}[\underline{X}] \mid f(\underline{x}) \geq 0 \ \forall \ \underline{x} \in K \Big\}$

Note that $T_S \subseteq \mathrm{Psd}(K_S)$.

Question. If $f \in \mathrm{Psd}(K_S)$, then does $f \in T_S$?

Answer. No.

But there is a connection of f with T_S (which will become clear through the Positivstellensatz in the next lecture).