REAL ALGEBRAIC GEOMETRY LECTURE NOTES (14: 01/12/2009 - BEARBEITET 08/12/2022)

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THE TARSKI-SEIDENBERG PRINCIPLE

Main Proposition. Let $f_i(\underline{T}, X) := h_{i,m_i}(\underline{T})X^{m_i} + \ldots + h_{i,0}(\underline{T})$ for $i = 1, \ldots, s$ be a sequence of polynomials in $n+1$ variables with coefficients in $\mathbb{Z}$, and let $m := max\{m_i | i = 1, \ldots, s\}$. Let W' be a subset of $W_{s,m}$. Then there exists a boolean combination $B(\underline{T}) = S_1(\underline{T}) \vee \ldots \vee S_p(\underline{T})$ of polynomial equations and inequalities in the variables $\underline{\mathrm{T}}$ with coefficients in $\mathbb{Z}$, such that, for every real closed field R and every $\underline{t} \in R^n$, we have

$$SIGN_R\big(f_1(\underline{t}, X), \ldots, f_s(\underline{t}, X)\big) \in W' \Leftrightarrow B(\underline{t}) \text{ holds true in } R.$$

Proof. Without loss of generality, we assume that none of $f_1, \ldots, f_s$ is identically zero and that $h_{i,m_i}(\underline{T})$ is not identically zero for $i = 1, \ldots, s$. To every sequence of polynomials $(f_1, \ldots, f_s)$ associate the s-tuple $(m_1, \ldots, m_s)$, where $deg(f_i) = m_i$. We compare these finite sequences by defining a strict order as follows:

$$\sigma := (m_1', \ldots, m_t') \prec \tau := (m_1, \ldots, m_s)$$

if there exists $p \in \mathbb{N}$ such that, for every $q > p$,
-the number of times q appears in σ = the number of times q appears in τ, and
-the number of times p appears in $\sigma <$ the number of times p appears in τ.

This order $\prec$ is a total order [1] on the set of finite sequences.

Example: let $m = \max(\{m_1, \ldots, m_s\}) = m_s$ (say), σ and τ be the sequence of degrees of the sequences $(f_1, \ldots, f_{s-1}, f_s', g_1, \ldots, g_s)$ and $(f_1, \ldots, f_{s-1}, f_s)$ respectively, i.e.
$\sigma \rightsquigarrow (f_1, \ldots, f_{s-1}, f_s', g_1, \ldots, g_s)$,
$\tau \rightsquigarrow (f_1, \ldots, f_{s-1}, f_s)$

[1] This was a mistake in the book *Real Algebraic Geometry* of J. Bochnak, M. Coste, M.-F. Roy. For corrected argument, see Appendix I following this proof.

then $\sigma \prec \tau$.

Let $m = \max\{m_1, \ldots, m_s\}$.

In particular using $p = m$ we have:

$\big(deg(f_1), \ldots, deg(f_{s-1}), deg(f_s^{'}), deg(g_1), \ldots, deg(g_s)\big) \prec \big(deg(f_1), \ldots, deg(f_s)\big)$.

If $\underline{m = 0}$, then there is nothing to show, since $SIGN_R\big(f_1(\underline{t}, X), \ldots, f_s(\underline{t}, X)\big) = SIGN_R\big(h_{1,0}(\underline{t}), \ldots, h_{s,0}(\underline{t})\big)$ [the list of signs of "constant terms"].

Suppose that $\underline{m \geq 1}$ and $m_s = m = max\{m_1, \ldots, m_s\}$. Let $W^{''} \subset W_{2s,m}$ be the inverse image of $W^{'} \subset W_{s,m}$ under the mapping φ (as in main lemma). Set $W^{''} = \big\{SIGN_R(f_1, \ldots, f_{s-1}, f_s^{'}, g_1, \ldots, g_s) \mid SIGN_R(f_1, \ldots, f_s) \in W^{'}\big\}$.

-**Case 1.** By the main lemma, for every real closed field R and for every $\underline{t} \in R^n$ such that $h_{i,m_i}(\underline{t}) \neq 0$ for $i = 1, \ldots, s$, we have

$$SIGN_R\big(f_1(\underline{t}, X), \ldots, f_s(\underline{t}, X)\big) \in W^{'}$$

$$\Leftrightarrow$$

$$SIGN_R\big(f_1(\underline{t}, X), \ldots, f_{s-1}(\underline{t}, X), f_s^{'}(\underline{t}, X), g_1(\underline{t}, X), \ldots, g_s(\underline{t}, X)\big) \in W^{''},$$

where $f_s^{'}$ is the derivative of f_s with respect to X, and $g_1, \ldots, g_s$ are the remainders of the euclidean division (with respect to X) of f_s by $f_1, \ldots, f_{s-1}, f_s^{'}$, respectively (multiplied by appropriate even powers of $h_{1,m_1}, \ldots, h_{s,m_s}$, respectively, to clear the denominators).
Now, the sequence of degrees in X of $f_1, \ldots, f_{s-1}, f_s^{'}, g_1, \ldots, g_s$ is smaller than [the sequence of degrees in X of $f_1, \ldots, f_s$ i.e.] $(m_1, \ldots, m_s)$ w.r.t. the order $\prec$.

-**Case 2.** At least one of $h_{i,m_i}(\underline{t})$ is zero
In this case we can truncate the corresponding polynomial f_i and obtain a sequence of polynomials, whose sequence of degrees in X is smaller than $(m_1, \ldots, m_s)$ w.r.t. the order $\prec$.

This completes the proof of main propostion and also proves the Tarski-Seidenberg principle. □□