REAL ALGEBRAIC GEOMETRY LECTURE NOTES (12: 26/11/2009 - BEARBEITET 01/12/2022)

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Contents

1. Quantifier eliminaton for the theory of real closed fields

We recall from last lecture the definition of first order formulas in the language of real closed field:

Definition 1.1. A **first order formula in the language of real closed fields** is obtained as follows recursively:

(1) if $f(\underline{x}) \in \mathbb{Q}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$, $n \geqslant 1$, then
$$f(\underline{\mathrm{x}}) \geqslant 0,\ f(\underline{\mathrm{x}}) > 0,\ f(\underline{\mathrm{x}}) = 0,\ f(\underline{x}) \neq 0$$
are first order formulas (with free variables $\underline{\mathrm{x}} = (\mathrm{x}_1, \ldots, \mathrm{x}_n)$);

(2) if Φ and Ψ are first order formulas, then
$$\Phi \wedge \Psi, \quad \Phi \vee \Psi, \quad \neg\Phi$$
are also first order formulas (with free variables given by the union of the free variables of Φ and the free variables of Ψ);

(3) if Φ is a first order formula then
$$\exists\, \mathrm{x}\, \Phi \quad \text{and} \quad \forall \mathrm{x}\, \Phi$$
are first order formulas (with same free variables as Φ minus $\{x\}$).

The formulas obtained using just (1) and (2) are called **quantifier free**.

Definition 1.2. Let $\Phi(\mathrm{x}_1, \ldots, \mathrm{x}_n)$ and $\Psi(\mathrm{x}_1, \ldots, \mathrm{x}_n)$ be first order formulas in the language of real closed fields with free variables contained in $\{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$. We say that $\Phi(\underline{\mathrm{x}})$ and $\Psi(\underline{\mathrm{x}})$ are **equivalent** if for every real closed field R and every $\underline{r} \in R^n$,
$$\Phi(\underline{r}) \text{ holds in } R \iff \Psi(\underline{r}) \text{ holds in } R.$$

If Φ and Ψ are equivalent, we write $\Phi \sim \Psi$.

Remark 1.3. (Normal form of quantifier free formulas). Every quantifier free formula is equivalent to a finite disjunction of finite conjuctions of formulas obtained using construction (1).

Proof. Like showing that every semialgebraic subset of R^n is a finite union (= finite disjunction) of basic semialgebraic sets (= finite conjuction of formulas of type (1)). □

Theorem 1.4. *(Tarski's quantifier elimination theorem for real closed fields). Every first order formula in the language of real closed fields is equivalent to a quantifier free formula.*

Proof. Since all formulas of type (1) are quantifier free, it suffices to show that

$\mathcal{C}$:= the set of first order formulas which are equivalent to quantifier free formulas

is closed under constructions of (2) and (3).

Closure under 2. If $\Phi \sim \Phi'$ and $\Psi \sim \Psi'$, then

$$\begin{array}{rcl} \Phi \vee \Psi & \sim & \Phi' \vee \Psi' \\ \Phi \wedge \Psi & \sim & \Phi' \wedge \Psi' \\ \neg\,\Phi & \sim & \neg\,\Phi'. \end{array}$$

Closure under 3. It is enough to consider $\exists\,\mathrm{x}\,\Phi$, because

$$\forall\,\mathrm{x}\,\Phi \;\leftrightarrow\; \neg\,\exists\,x\;(\neg\Phi).$$

We claim that if Φ is equivalent to a quantifier free formula then $\exists\,\mathrm{x}\,\Phi$ is equivalent to a quantifier free formula. Since

$$\exists\,\mathrm{x}\,(\Phi_1 \vee \cdots \vee \Phi_k) \;\sim\; (\exists\,\mathrm{x}\,\Phi_1) \vee \cdots \vee (\exists\,\mathrm{x}\,\Phi_k),$$

using the normal form of quantifier free formulas (Remark 1.3), we can assume that Φ is a finite conjunction of polynomial equations and inequalities (i.e. a system $S(\underline{T}, \mathrm{x})$).

Applying Tarski-Seidenberg's Theorem:

$$\exists\,\underline{\mathrm{x}}\; S(\underline{T}; \underline{\mathrm{x}}) \;\Leftrightarrow\; \bigvee_{i=1}^{l} S_i(\underline{t}),$$

there exist finitely many finite conjunctions of polynomial equalities and inequalities $\vartheta_1, \ldots, \vartheta_l$ (corresponding to the systems $S_1(\underline{T}), \ldots, S_l(\underline{T})$) such that

$$\exists\,\underline{\mathrm{x}}\;\Phi \;\sim\; \vartheta_1 \vee \cdots \vee \vartheta_l.$$

□

2. Definable sets

Definition 2.1. Let $\Phi(\underline{T}, \underline{X})$ a first order formula with free variables $T_1, \ldots, T_m, X_1, \ldots, X_n$. Let R be a real closed field and $\underline{t} \in R^m$. Then $\Phi(\underline{t}, \underline{X})$ is a **first order formula with parameters** in R, and $t_1, \ldots, t_m$ are called the parameters.

Definition 2.2. Let R be a real closed field, $n \geqslant 1$. A subset $A \subseteq R^n$ is said to be **definable** (**with parameters** from R) in R if there is a first order formula $\Phi(\underline{t}, \underline{X})$ with parameters $\underline{t} \in R^m$ and free variables $\underline{X} = (X_1, \ldots, X_n)$, such that

$$A = \{\underline{r} \in R^n : \Phi(\underline{t}, \underline{r}) \text{ is true in } R\}.$$

Corollary 2.3. *For any real closed field R the class of definable sets (with parameters) in R coincides with the class of semialgebraic sets.*