REAL ALGEBRAIC GEOMETRY LECTURE NOTES (10: 20/11/2009 - BEARBEITET 24/11/2022)

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Contents

1. Homomorphism Theorems

Theorem 1.1. *(Homomorphism Theorem I) Let $R \subseteq R_1$ be real closed fields and $I \subset R[\underline{x}]$ an ideal. Then*

$$\exists \text{ } R\text{-alg. hom. } \varphi\colon \frac{R[\underline{x}]}{I} \longrightarrow R_1 \;\Rightarrow\; \exists \text{ } R\text{-alg. hom. } \psi\colon \frac{R[\underline{x}]}{I} \longrightarrow R.$$

Corollary 1.2. *(Homomorphism Theorem II) Suppose R and R_1 are real closed fields, $R \subseteq R_1$. Let A be a finitely generated R-algebra. If there is an R-algebra homomorphism*

$$\varphi\colon A \longrightarrow R_1$$

then there is an R-algebra homomorphism

$$\psi\colon A \longrightarrow R.$$

Proof. We want to use Homomorphism Theorem I. For this we just prove the following:

Claim 1.3. *A is a finitely generated R-algebra if and only if there is a surjective R-algebra homomorphism $\vartheta\colon R[\mathrm{x}_1, \ldots, \mathrm{x}_n] \longrightarrow A$ (for some $n \in \mathbb{N}$).*

Proof.

(⇒) Let A be a finitely generated R-algebra, say with generators $r_1, \ldots, r_n$. Define $\vartheta\colon R[\mathrm{x}_1, \ldots, \mathrm{x}_n] \longrightarrow A$ by setting $\vartheta(\mathrm{x}_i) := r_i$ for every $i = 1, \ldots, n$, and $\vartheta(a) := a$ for every $a \in R$.

(⇐) Given a surjective homomorphism $\vartheta\colon R[\mathrm{x}_1, \ldots, \mathrm{x}_n] \longrightarrow A$ set $r_i := \vartheta(\mathrm{x}_i) \in A$ for every $i = 1, \ldots, n$. Then $\{r_1, \ldots, r_n\}$ generate A over R.

□

So we get $A \cong R[\underline{\mathrm{x}}]/I$ with $I = \ker \vartheta$.

□

We can see that Homomorphism Theorem II implies T-T-III:

Let $R \subset R_1$ be real closed fields. $S(\underline{X})$ with coefficients in R has a solution $\underline{x} \in R_1^n$ if and only if it has a solution $\underline{x} \in R^n$.

We first need the following:

Proposition 1.4. *Let*

$$S(\underline{\mathrm{x}}) := \begin{cases} f_1(\underline{\mathrm{x}}) \lhd_1 0 \\ \quad \vdots \\ f_k(\underline{\mathrm{x}}) \lhd_k 0 \end{cases}$$

be a system with coefficients in R*, where* $\lhd_i \in \{\geqslant, >, =, \neq\}$*. Then* $S(\underline{\mathrm{x}})$ *can be written as a system of the form*

$$\sigma(\underline{\mathrm{x}}) := \begin{cases} g_1(\underline{\mathrm{x}}) \geqslant 0 \\ \quad \vdots \\ g_s(\underline{\mathrm{x}}) \geqslant 0 \\ g(\underline{\mathrm{x}}) \neq 0 \end{cases}$$

for some $g_1, \ldots, g_s, g \in R[\underline{\mathrm{x}}]$*.*

Proof.

- Replace each equality in the original system by a pair of inequalities:

$$f_i = 0 \ \Leftrightarrow \ \begin{cases} f_i \geqslant 0 \\ -f_i \geqslant 0 \end{cases}$$

- Replace each strict inequality

$$f_i > 0 \ \text{ by } \ \begin{cases} f_i \geqslant 0 \\ f_i \neq 0 \end{cases}$$

- Finally collect all inequalities $f_i \neq 0$, $i = 1, \ldots, t$ as

$$g := \prod_{i=1}^{t} f_i \neq 0.$$

□

Now we show that Homomorphism Theorem II implies T-T-III:

Proof. Let $R \subseteq R_1$ and let $S(\underline{\mathrm{x}})$ be a system with coefficients in R:

$$S(\underline{\mathrm{x}}) := \begin{cases} f_1(\underline{\mathrm{x}}) \lhd_1 0 \\ \quad \vdots \\ f_k(\underline{\mathrm{x}}) \lhd_k 0 \end{cases}$$

Rewrite it as

$$S(\underline{x}) := \begin{cases} f_1(\underline{x}) \geqslant 0 \\ \vdots \\ f_k(\underline{x}) \geqslant 0 \\ g(\underline{x}) \neq 0 \end{cases}$$

with $f_i(\underline{x}), g(\underline{x}) \in R[x_1, \dots, x_n]$.

Suppose $\underline{x} \in R_1^n$ is a solution of $S(\underline{x})$. Consider

$$A := \frac{R[X_1, \dots, X_n, Y_1, \dots, Y_k, Z]}{\langle Y_1^2 - f_1, \dots, Y_k^2 - f_k; gZ - 1\rangle},$$

which is a finitely generated R-algebra. Consider the R-algebra homomorphism φ such that

$$\begin{aligned} \varphi\colon A &\longrightarrow R_1 \\ \bar{X}_i &\mapsto x_i \\ \bar{Y}_j &\mapsto \sqrt{f_j(\underline{x})} \\ \bar{Z} &\mapsto 1/g(\underline{x}). \end{aligned}$$

By Homomorphism Theorem II there is an R-algebra homomorphism $\psi\colon A \longrightarrow R$. Then $\psi(\bar{X}_1), \dots, \psi(\bar{X}_n)$ is the required solution in R^n.

□

2. Hilbert's 17^{th} problem

Definition 2.1. Let R be a real closed field. We say that a polynomial $f(\underline{x}) \in R[\underline{x}]$ is **positive semi-definite** if $f(x_1, \dots, x_n) \geqslant 0 \ \forall \, (x_1, \dots, x_n) \in R^n$. We write $f \geqslant 0$.

We know that

$$f \in \sum R[\underline{x}]^2 \ \Rightarrow \ f \geqslant 0.$$

Now take $R = \mathbb{R}$. Conversely, for any $f \in \mathbb{R}[x]$ is it true that

$$f \geqslant 0 \text{ on } \mathbb{R}^n \overset{?}{\Rightarrow} f \in \sum \mathbb{R}(\underline{x})^2. \qquad \textbf{(Hilbert's } 17^{th} \textbf{ problem)}.$$

Remark 2.2.

(1) Hilbert knew that the answer is NO to the more natural question

$$f \in \mathbb{R}[\underline{x}],\ f \geqslant 0 \text{ on } \mathbb{R}^n \ \Rightarrow \ f \in \sum \mathbb{R}[\underline{x}]^2 \ ?$$

(2) If $n = 1$ then indeed $f \geqslant 0$ on $\mathbb{R} \ \Rightarrow \ f = f_1^2 + f_2^2$.

(3) More generally Hilbert showed that:

Set $P_{d,n}$:= the set of homogeneous polynomials of degree d in n-variables which are positive semi-definite

and set $\sum_{d,n}$:= the subset of $P_{d,n}$ consisting of sums of squares.

Then

$P_{d,n} = \sum_{d,n} \iff n \leqslant 2$ or $d = 2$ or ($n = 3$ and $d = 4$).

Note: only d even is interesting because

Lemma 2.3. $0 \neq f \in \sum \mathbb{R}[\underline{x}]^2 \Rightarrow \deg(f)$ *is even. More precisely, if* $f = \sum_{i=1}^{k} f_i^2$, *with* $f_i \in \mathbb{R}[\underline{x}]$ $f_i \neq 0$, *then* $\deg(f) = 2\max\{\deg(f_i) : i = 1, \ldots, k\}$.

Hilbert knew that $P_{6,3} \setminus \sum_{6,3} \neq \emptyset$.

The first example was given by Motzkin 1967:

$$m(X, Y, Z) = X^6 + Y^4Z^2 + Y^2Z^4 - 3X^2Y^2Z^2.$$

Theorem 2.4. *(Artin, 1927) Let R be a real closed field and $f \in R[\underline{x}]$, $f \geqslant 0$ on R^n. Then $f \in \sum R(\underline{x})^2$.*

Proof. Set $F = R(\underline{x})$ and $T = \sum F^2 = \sum R(\underline{x})^2$. Note that since $R(\underline{x})$ is real, $\sum F^2$ is a proper preordering.

We want to show:

$$f \notin T \ \Rightarrow\ \exists\, \underline{x} \in R^n : f(\underline{x}) < 0.$$

Since $f \in F \setminus T$, by Zorn's Lemma there is a preordering $P \supseteq T$ of F which is maximal for the property that $f \notin P$. Then P is an ordering of F (see proof of Crucial Lemma 2.1 of Lecture 3).

Let $\leqslant_P$ be the ordering such that $(F, \leqslant_P)$ is an ordered field extension of the real closed field R (since R is a real closed field, it is uniquely ordered and we know that $(F, \leqslant_P)$ is an ordered field extension). By construction $f \notin P$ so $f < 0$. Consider the system

$$S(\underline{x}) := \Big\{ f(\underline{x}) < 0, \qquad f(\underline{x}) \in R[\underline{x}].$$

This system has a solution in $F = R(\underline{x})$, namely

$$\underline{X} = (X_1, \ldots, X_n) \qquad X_i \in R(\underline{x}) = F.$$

thus by T-T-IV $\exists\, \underline{x} \in R^n$ with $f(\underline{x}) < 0$. $\square$