REAL ALGEBRAIC GEOMETRY LECTURE NOTES (08: 12/11/2009 - BEARBEITET 17/11/2022)

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Contents

1. Real closure

Definition 1.1. Let (K, P) be an ordered field. R is a real closure of (K, P) if

(1) R is real closed,
(2) $R \supseteq K$, $R \,|\, K$ is algebraic,
(3) $P = \sum R^2 \cap K$ (i.e. the order on K is the restriction of the unique order R to K).

Theorem 1.2. *Every ordered field (K, P) has a real closure.*

Proof. Apply Zorn's Lemma and Proposition 5.1.1(ii) to

$$\mathcal{L} := \{(L, Q) : L \,|\, K \text{ algebraic},\ Q \cap K = P\}.$$

□

Proposition 1.3. *(Corollary to Sturm's Theorem) Let K be a field. Let R_1, R_2 be two real closed fields such that*

$$K \subseteq R_1 \quad \textit{and} \quad K \subseteq R_2$$

with

$$P := K \ \cap \ \sum R_1^2 = K \ \cap \ \sum R_2^2$$

(i.e. R_1 and R_2 induce the same ordering P on K).

Let $f(\mathrm{x}) \in K[\mathrm{x}]$; then the number of roots of $f(\mathrm{x})$ in R_1 is equal to the number of roots of $f(\mathrm{x})$ in R_2.

2. Order preserving extensions

Proposition 2.1. *Let (K,P) be an ordered field. Let R be a real closed field containing (K,P). Let $K \subseteq L \subseteq R$ be such that $[L:K] < \infty$. Let S be a real closed field with*

$$\varphi\colon (K,P) \hookrightarrow (S, \sum S^2)$$

an order preserving embedding. Then φ extends to an order preserving embedding

$$\psi\colon (L, \sum R^2 \cap L) \hookrightarrow (S, \sum S^2).$$

Proof. We recall that if (K,P) and (L,Q) are ordered fields, a field homomorphism $\varphi\colon K \longrightarrow L$ is called **order preserving** with respect to P and Q if $\varphi(P) \subseteq Q$ (equivalently $P = \varphi^{-1}(Q)$).

By the Theorem of the Primitive Element $L = K(\alpha)$.

Consider $f = MinPol(\alpha \,|\, K)$. Since $\alpha \in R$, $\varphi(f)$ has at least one root β in S by Proposition 1.3

$$L := K(\alpha) \quad \overset{\psi}{\longleftrightarrow} \quad \varphi(K)(\beta),$$

so there is at least one extension of φ from K to L.

Let $\psi_1, \dots, \psi_r$ all such extensions of φ to $L = K(\alpha)$, and for a contradiction assume that none of them is order preserving with respect to $Q = L \cap \sum R^2$. Then $\exists\, b_1, \dots, b_r \in L$, $b_i > 0$ (in R) and $\psi_i(b_i) < 0$ (in S) $\forall\, i = 1, \dots, r$.

Consider $L' := L(\sqrt{b_1}, \dots, \sqrt{b_r}) \subset R$. Since $[L:K] < \infty$, also $[L':K] < \infty$.

So let τ be an extension of φ from K to L'. In particular $\tau_{|_L}$ is one of the ψ_i's. Say $\tau_{|_L} = \psi_1$.

Now compute for $b_1 \in L$,

$$\psi_1(b_1) = \tau(b_1) = \tau((\sqrt{b_1})^2) = (\tau(\sqrt{b_1}))^2 \in \sum S^2,$$

in contradiction with the fact that $\psi_1(b_1) < 0$.

□

Theorem 2.2. *Let (K,P) be an ordered field and $(R, \sum R^2)$ be a real closure of (K,P). Let $(S, \sum S^2)$ be a real closed field and assume that*

$$\varphi\colon (K,P) \hookrightarrow (S, \sum S^2)$$

is an order preserving embedding. Then φ has a uniquely determined extension

$$\psi\colon (R, \sum R^2) \hookrightarrow (S, \sum S^2).$$

Proof. Consider

$$\mathcal{L} := \{(L,\psi) :\ K \subset L \subset R;\ \psi\colon L \hookrightarrow S,\ \psi_{|_K} = \varphi\}.$$

Let (L, ψ) be a maximal element. Then by Proposition 2.1 we must have $L = R$.

Therefore we have an order preserving embedding ψ of R extending φ

$$\psi\colon R \hookrightarrow S.$$

We want to prove that ψ is unique. We show that $\psi(\alpha) \in S$ is uniquely determined for every $\alpha \in R$.

Let $f = MinPol(\alpha \,|\, K)$ and let $\alpha_1 < \cdots < \alpha_r$ all the real roots of f in R. Let $\beta_1 < \cdots < \beta_r$ be all the real roots of $\psi(f)$ in S. Since $\psi\colon R \hookrightarrow S$ is order preserving, we must have $\psi(\alpha_i) = \beta_i$ for every $i = 1, \ldots, r$. In particular $\alpha = \alpha_j$ for some $1 \leqslant j \leqslant r$ and $\psi(\alpha) = \beta_j \in S$. □

Corollary 2.3. *Let (K, P) be an ordered field, R_1, R_2 two real closures of (K, P). Then there exists a unique*

$$\varphi\colon R_1 \longrightarrow R_2$$

K-isomorphism (i.e. with $\varphi_{|_K} = id$).

Corollary 2.4. *Let R be a real closure of (K, P). Then the only K-automorphism of R is the identity.*

Corollary 2.5. *Let R be a real closed field, $K \subseteq R$ a subfield. Set $P := K \cap \sum R^2$ the induced order. Then*

$$K^{ralg} = \{\alpha \in R : \alpha \text{ is algebraic over } K\}$$

is relatively algebraic closed in R and is a real closure of (K, P).

Proof. It is enough to show that K^{ralg} is real closed.

K^{ralg} is real because $Q := K^{ralg} \cap \sum R^2$ is an induced ordering.

Let $a \in Q$, $a = b^2$, $b \in R$. So $p(\mathrm{x}) = \mathrm{x}^2 - a \in K^{ralg}[\mathrm{x}]$ has a root in R.

One can see that b is algebraic over K (so $b \in K^{ralg}$).

Similarly one shows that every odd polynomial with coefficients in K^{ralg} has a root in K^{ralg}. □

Corollary 2.6. *Let (K, P) be an ordered field, S a real closed field and $\varphi\colon (K, P) \hookrightarrow S$ an order preserving embedding. Let $L \,|\, K$ an algebraic extension. Then there is a bijective correspondence*

$$\{\text{extensions } \psi\colon L \to S \text{ of } \varphi\} \xrightarrow{\mathcal{E}} \{\text{extensions } Q \text{ of } P \text{ to } L\}$$
$$\psi \quad \mapsto \quad \psi^{-1}(\textstyle\sum S^2)$$

Proof.

($\Rightarrow$) Let $\psi\colon L \hookrightarrow S$ an extension of φ. Then indeed $Q := \psi^{-1}(\sum S^2)$ is an ordering on L. Furthermore $\psi^{-1}(\sum S^2) \cap K = \varphi^{-1}(\sum S^2) = P$. So the extension ψ induces the extension Q.

($\Leftarrow$) Conversely assume that Q is an extension of P from K to L ($Q \cap K = P$). Note that if R is a real closure of (L, Q) then R is a real closure of (K, P) as well.

Now apply Theorem 2.2 to extend φ to $\sigma\colon R \to S$. Set $\psi := \sigma_{|L}$ which is order preserving with respect to Q.

So the map $\mathcal{E}$ is well-defined and surjective. To see that it is also injective, assume

$$\psi_1\colon L \longrightarrow S, \quad \psi_2\colon L \longrightarrow S, \quad {\psi_1}_{|K} = {\psi_2}_{|K} = \varphi$$

which induce the same order

$$Q = \psi_1^{-1}(\sum S^2) = \psi_2^{-1}(\sum S^2)$$

on L. Let R be the real closure of (L, Q). Apply Theorem 2.2 to ψ_1 and ψ_2 to get uniquely determined extensions

$$\sigma_1\colon R \longrightarrow S, \quad \sigma_2\colon R \longrightarrow S,$$

of ψ_1 and ψ_2 respectively.

But now ${\sigma_1}_{|K} = {\sigma_2}_{|K} = \varphi$. By the uniqueness part of Theorem 2.2 we get $\sigma_1 = \sigma_2$ and a fortiori $\psi_1 = \psi_2$.

$\square$

Corollary 2.7. *Let (K, P) be an ordered field, R a real closure, $[L : K] < \infty$. Let $L = K(\alpha)$, $f = MinPol(\alpha \,|\, K)$. Then there is a bijection*

$$\{\textit{roots of } f \textit{ in } R\} \longrightarrow \{\textit{extensions } Q \textit{ of } P \textit{ to } L\}.$$

Proof. If β is a root we consider the K-embedding

$$\varphi_\alpha\colon L \hookrightarrow R$$

such that $\varphi_\alpha(\alpha) = \beta$. Set $Q := \varphi^{-1}(\sum R^2)$ ordering on L extending P. $\square$

Example 2.8. $K = \mathbb{Q}(\sqrt{2})$ has 2 orderings $P_1 \neq P_2$, with $\sqrt{2} \in P_1$, $\sqrt{2} \notin P_2$. The Minimum Polynomial of $\sqrt{2}$ over $\mathbb{Q}$ is $p(\mathrm{x}) = \mathrm{x}^2 - 2$.