REAL ALGEBRAIC GEOMETRY LECTURE NOTES (07: 10/11/09 - BEARBEITET 15/11/2022)

SALMA KUHLMANN

Contents

Let R be a real closed field.

1. Sturm's Theorem

Definition 1.1.

(i) Let $f \in R[\mathrm{x}]$ be a non-constant polynomial, $\deg(f) \geqslant 1$. The **Sturm sequence** of f is defined recursively as the sequence $(f_0, \ldots, f_r)$ of polynomials in $R[\mathrm{x}]$ such that:

$$\begin{aligned} f_0 &:= f, \qquad f_1 := f' \qquad \textit{and} \\ f_0 &= f_1 q_1 - f_2 \\ f_1 &= f_2 q_2 - f_3 \\ &\ldots \\ f_{i-1} &= f_i q_i - f_{i+1} \\ &\ldots \\ f_{r-2} &= f_{r-1} q_{r-1} - f_r \\ f_{r-1} &= f_r q_r, \end{aligned}$$

where $f_i, q_i \in R[\mathrm{x}]$, $f_i \neq 0$ and $\deg(f_i) < \deg(f_{i-1})$ r, f_i, q_i uniquely determined.

(ii) Let $x \in R$. Set

$$V_f(x) := \mathrm{Var}(f_0(x), \ldots, f_r(x)).$$

We recall that after we have removed all zero's by the sequence $(c_1, \ldots, c_n)$, we defined $\mathrm{Var}(c_1, \ldots, c_n)$ as the number of changes of sign in $(c_1, \ldots, c_n)$, i.e.

$$\mathrm{Var}(c_1, \ldots, c_n) = |\{i \in \{1, \ldots, n-1\} : c_i c_{i+1} < 0\}|.$$

Theorem 1.2. *(Sturm 1829). Let $a, b \in R$, $a < b$, $f(a)f(b) \neq 0$. Then*

$$|\{c : a \leqslant c \leqslant b\,,\ f(c) = 0\}| \ = \ V_f(a) - V_f(b).$$

Proof. For the proof we study the function $V_f(x)$, $x \in R$, locally constant except around finitely many roots for $f_0, \dots, f_r$.

(1) Suppose $\gcd(f_0, f_1) = 1$.

(2) Hilfslemma (ÜA) Let $c \in R$ be a root of f_0. Then $\exists\,\delta$ such that

$$|x-c| < \delta \;\Rightarrow\; \operatorname{sign}(f_0(x)f_1(x)) = \operatorname{sign}(x-c) = \begin{cases} -1 & \text{if } x < c \\ 0 & \text{if } x = c \\ 1 & \text{if } x > c. \end{cases}$$

(3) $\forall\, i \in \{1, \dots, r-1\}$: $\gcd(f_{i-1}, f_i) = 1$ and

$$f_{i-1} = q_i f_i - f_{i+1}, \qquad \text{with } f_{i+1} \neq 0.$$

So if $f_i(c) = 0$ then

$$f_{i-1}(c)f_{i+1}(c) < 0.$$

(4) Let $f_i(c) = 0$ for some $i \in \{0, \dots, r-1\}$. Then $f_{i+1}(c) \neq 0$ (so $\operatorname{sign}(f_{i+1}(c)) = \pm 1$).

We shall now compare for $f_i(c) = 0$,

$$\operatorname{sign}(f_i(x)) \qquad \operatorname{sign}(f_{i+1}(x))$$

for $|x-c| < \delta$ and count.

We first examine the case $i = 0$.

Observe that $\operatorname{sign}(f_1(x)) \neq 0\ \forall\, x$ such that $|x-c| < \delta$ because of Hilfslemma. So in particular $\operatorname{sign}(f_1(x))$ is constant for $|x-c| < \delta$ and it is equal to $\operatorname{sign}(f_1(c)$:

	$x \to c_-$	$x = c$	$x \to c_+$
$f_0(x)$	$-\operatorname{sign}(f_1(c))$	0	$\operatorname{sign}(f_1(c))$
$f_1(x)$	$\operatorname{sign}(f_1(c))$	$\operatorname{sign}(f_1(c))$	$\operatorname{sign}(f_1(c))$
contribution to $V_f(x)$	1	0	0

Now consider $i \in \{1, \dots, r-1\}$ and use (3), i.e.

$$f_i(d) = 0 \implies f_{i-1}(d)f_{i+1}(d) < 0:$$

	$x \to d_-$	$x = d$	$x \to d_+$
$f_{i-1}(x)$	$-\operatorname{sign}(f_{i+1}(d))$	$-\operatorname{sign}(f_{i+1}(d))$	$-\operatorname{sign}(f_{i+1}(d))$
$f_i(x)$		0	
$f_{i+1}(x)$	$\operatorname{sign}(f_{i+1}(d))$	$\operatorname{sign}(f_{i+1}(d))$	$\operatorname{sign}(f_{i+1}(d))$
contribution to $V_f(x)$	1	1	1

Therefore for $a < b$, $V_f(a) - V_f(b)$ is the number of roots of f in $]a, b[$.

Let us consider now the general case. Set

$$g_i := f_i/f_r \qquad i = 0, \dots, r.$$

The sequence of polynomials $(g_0, \dots, g_r)$ satisfies the previous conditions $(1)-(4)$. We can conclude by noticing that:

(i) $\mathrm{Var}(g_0(x), \dots, g_r(x)) = \mathrm{Var}(f_0(x), \dots, f_r(x))$ (because $f_i(x) = f_r(x)g_i(x)$),

(ii) $f = f_0$ and $g_0 = f/f_r$ have the same zeros ($f_r = \gcd(f, f')$, so $g = f/f_r$ has only simple roots, whereas f has roots with multiplicities.)

□

For $i = 0, \dots, r$ set $d_i := \deg(f_i)$ and $\varphi_i :=$ the leading coefficient of f_i. Set

$$V_f(-\infty) := \mathrm{Var}((-1)^{d_0}\varphi_0,\, (-1)^{d_1}\varphi_1, \dots, (-1)^{d_r}\varphi_r)$$
$$V_f(+\infty) := \mathrm{Var}(\varphi_0, \varphi_1, \dots, \varphi_r).$$

Then we have:

Corollary 1.3. *The number of distinct roots of f is $V_f(-\infty) - V_f(+\infty)$.*