REAL ALGEBRAIC GEOMETRY LECTURE NOTES (06: 05/11/2009 - BEARBEITET 10/11/2022)

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Contents

Let R be a real closed field (for all this lecture).

1. Counting roots in an interval

Definition 1.1. Let $f(\mathrm{x}) \in R[\mathrm{x}]$, $a \in R$,

$$f(\mathrm{x}) = (\mathrm{x} - a)^m h(\mathrm{x})$$

with $m \in \mathbb{N}$, $m \geqslant 1$ and $h(a) \neq 0$ (i.e. $(\mathrm{x} - a)$ is not a factor of $h(\mathrm{x})$).

We say that m is the **multiplicity** (*Vielfachheit*) of f at a.

Corollary 1.2. *(Generalized Intermediate Value Theorem: Verstärkung Zwischenwertsatz). Let $f(\mathrm{x}) \in R[\mathrm{x}]$; $a, b \in R$, $a < b$, $f(a)f(b) < 0$ (i.e. $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$). Then the number of roots of $f(\mathrm{x})$ counting multiplicities in the interval $]a, b[\subseteq R$ is odd (in particular, f has a root in $]a, b[$).*

Proof. By Corollary 3.1 of 5th lecture (3/11/09), we can write

$$f(\mathrm{x}) = \prod_{i=1}^{n} (\mathrm{x} - c_i)^{m_i} g(\mathrm{x})$$

with $g(\mathrm{x}) = dq(\mathrm{x})$, where $d \in R$ is the leading coefficient of $f(\mathrm{x})$ and $q(\mathrm{x})$ is the product of the irreducible quadratic factors of $f(\mathrm{x})$.

Note that $g(\mathrm{x})$ has constant sign on R (i.e. $g(r) > 0 \ \forall\, r \in R$ or $g(r) < 0 \ \forall\, r \in R$). Without loss of generality, we can suppose $d = 1$ (and so $g(\mathrm{x})$ is positive everywhere).

Set $\forall\, i = 1, \ldots, n$

$$\begin{cases} L_i(\mathrm{x}) := (\mathrm{x} - c_i)^{m_i} \\ l_i(\mathrm{x}) := \mathrm{x} - c_i. \end{cases}$$

If $l_i(a)l_i(b) < 0$, then we must have $l_i(a) < 0 < l_i(b)$. Note that $L_i(a)L_i(b) < 0$ if and only if $l_i(a)l_i(b) < 0$ and m_i is odd.

In particular if $L_i(a)L_i(b) < 0$, then we must have $L_i(a) < 0 < L_i(b)$ as well.

Let us count the number of distinct $i \in \{1, \dots, n\}$ for which $L_i(a) < 0 < L_i(b)$. We claim that this number must be odd. If not, we get an even number of i such that $L_i(a)L_i(b) < 0$, so their product would be positive, in contradiction with the fact that $f(a)f(b) < 0$.

Set

$$|\{i \in \{1, \dots, n\} : L_i(a) < 0 < L_i(b)\}| = M \geqslant 1 \qquad \text{odd.}$$

Say these are $L_1, \dots, L_M$. So the total number of roots of f in $]a, b[$ counting multiplicity is

$$\sum := m_1 + \cdots + m_M.$$

Since m_i is odd $\forall\, i = 1, \dots, M$ and M is odd, it follows that $\sum$ is odd as well.

$\square$

2. Bounding the roots

Corollary 2.1. *Let* $f(\mathrm{x}) \in R[\mathrm{x}]$, $f(\mathrm{x}) = d\mathrm{x}^m + d_{m-1}\mathrm{x}^{m-1} + \cdots + d_0, d \neq 0.$ *Set*

$$D := 1 + \sum_{i=m-1}^{0} \left| \frac{d_i}{d} \right| \in R.$$

Then

(*i*) $a \in R,\ f(a) = 0 \ \Rightarrow\ |a| < D$;
(*i.e.* f *has no root in* $]-\infty, -D] \cup [D + \infty[$)

(*ii*) $y \in [D, +\infty[\ \Rightarrow\ \operatorname{sign}(f(y)) = \operatorname{sign}(d)$;

(*iii*) $y \in\ ]-\infty, -D[\ \Rightarrow\ \operatorname{sign}(f(y)) = (-1)^m \operatorname{sign}(d)$.

Proof. *W*log assume $\exists i$ such that $d_i \neq 0$.

(*i*) For every $i = 0, \dots, m-1$ set $b_i := \frac{d_i}{d}$ and compute for $|y| \geqslant D$:

$$f(y) = dy^m(1 + b_{m-1}y^{-1} + \cdots + b_0 y^{-m}).$$

Now

$$|b_{m-1}y^{-1} + \cdots + b_0 y^{-m}| \leqslant (|b_{m-1}| + \cdots + |b_0|)D^{-1} < 1$$

because $D > 1$, so $f(y) \neq 0$.

(*ii*) If $y \geqslant D$ then $f(y) = d \prod (y - a_i)^{m_i} q(y)$ where $\deg(q)$ is even and by (*i*), we have $|a_i| < D$, so $y - a_i > 0$.

(*iii*) If $y \leqslant -D$ then by (*i*), $(y - a_i)^{m_i} < 0$ if and only if m_i is odd. Moreover m is odd if and only if $\sum m_i$ is odd.

$\square$

Corollary 2.2. *(Rolle's Satz) Let $f(\mathrm{x}) \in R[\mathrm{x}]$, $a < b \in R$ such that $f(a) = f(b)$. Then there is $c \in R$, $a < c < b$ such that $f'(c) = 0$.*

Proof. We can suppose $f(a) = f(b) = 0$ (otherwise if $f(a) = f(b) = k \neq 0$, we can consider the polynomial $(f - k)(\mathrm{x})$).

We can also assume that $f(\mathrm{x})$ has no root in $]a, b[$. So

$$f(\mathrm{x}) = (\mathrm{x} - a)^m (\mathrm{x} - b)^n g(\mathrm{x}),$$

where $g(\mathrm{x})$ has no root in $[a, b]$, and by Corollary 1.2 (IVT) $g(\mathrm{x})$ has constant sign in $[a, b]$. Compute

$$f'(\mathrm{x}) = (\mathrm{x} - a)^{m-1} (\mathrm{x} - b)^{n-1} g_1(\mathrm{x}),$$

where

$$g_1(\mathrm{x}) := m(\mathrm{x} - b)g(\mathrm{x}) + n(\mathrm{x} - a)g(\mathrm{x}) + (\mathrm{x} - a)(\mathrm{x} - b)g'(\mathrm{x}).$$

Therefore

$$\begin{aligned} g_1(a) &= m(a - b)g(a) \\ g_1(b) &= n(b - a)g(b). \end{aligned}$$

Since $g_1(a)g_1(b) < 0$, by the Intermediate Value Theorem (1.2) $g_1(\mathrm{x})$ has a root in $]a, b[$ and so does $f'(\mathrm{x})$. □

Corollary 2.3. *(Mittelwertsatz: Mean Value Theorem) Let $f(\mathrm{x}) \in R[\mathrm{x}]$, $a < b \in R$. Then there is $c \in R$, $a < c < b$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We can apply Rolle's Theorem to

$$F(\mathrm{x}) := f(\mathrm{x}) - (\mathrm{x} - a)\frac{f(b) - f(a)}{b - a},$$

since $F(a) = F(b)$. □

Corollary 2.4. *(Monotonicity Theorem). Let $f(\mathrm{x}) \in R[\mathrm{x}]$, $a < b \in R$. If f' is positive (respectively negative) on $]a, b[$, then f is strictly increasing (respectively strictly decreasing) on $[a, b]$.*

Proof. If $a \leqslant a_1 < b_1 \leqslant b$, by the Mean Value Theorem there is some $c \in R$, $a_1 < c < b_1$ such that

$$f'(c) = \frac{f(b_1) - f(a_1)}{b_1 - a_1}.$$

□

3. Changes of sign

Definition 3.1.

(*i*) Let $(c_1, \dots, c_n)$ a finite sequence in R. An index $i \in \{1, \dots, n-1\}$ is a **change of sign** (*Vorzeichenwechsel*) if $c_i c_{i+1} < 0$.

(*ii*) Let $(c_1, \dots, c_n)$ a finite sequence in R. After we have removed all zero's by the sequence, we define

$$\begin{aligned} \mathrm{Var}(c_1, \dots, c_n) : &= |\{i \in \{1, \dots, n-1\} : i \text{ is a change of sign}\}| \\ &= |\{i \in \{1, \dots, n-1\} : c_i c_{i+1} < 0\}|. \end{aligned}$$

Theorem 3.2. *(Lemma von Descartes) Let* $f(\mathrm{x}) = a_n \mathrm{x}^n + \cdots + a_0 \in R[\mathrm{x}]$, $a_n \neq 0$. *Then*

$$|\{a \in R : a > 0 \text{ and } f(a) = 0\}| \;\leqslant\; \mathrm{Var}(a_n, \dots, a_1, a_0).$$

Proof. By induction on $n = \deg(f)$. The case $n = 1$ is obvious, so suppose $n > 1$. *W*log assume that $a_0 \neq 0$.

Let $r > 0$ be the smallest positive index such that $a_r \neq 0$. By induction applied to

$$f'(\mathrm{x}) = n a_n \mathrm{x}^{n-1} + \cdots + r a_r \mathrm{x}^{r-1} = x^{r-1} h(x) \text{ with } h(0) = a_r,$$

We know that there are at most $\mathrm{Var}(n a_n, \dots, r a_r) = \mathrm{Var}(a_n, \dots, a_r)$ many positive roots of f'. Set $c :=$ the smallest such positive root of f' (by convention $c := +\infty$ if none exists)

Apply Rolle's Theorem: f has at most $1 + \mathrm{Var}(a_n, \dots, a_r)$ positive roots. We consider the following two cases:

Case 1. If the number of positive roots of f is strictly less than $1 + \mathrm{Var}(a_n, \dots, a_r)$, then the number of positive roots of f is $\leqslant \mathrm{Var}(a_n, \dots, a_r) \leqslant \mathrm{Var}(a_n, \dots, a_r, a_0)$ and we are done.

Case 2. Assume f has exactly $1 + \mathrm{Var}(a_n, \dots, a_r)$ positive roots. We claim that in this case

$$1 + \mathrm{Var}(a_n, \dots, a_r) = \mathrm{Var}(a_n, \dots, a_r, a_0).$$

We observe that f has a root a in $]0, c[$.

For $0 < x < c$ we have that $\mathrm{sign}(f'(x)) = \mathrm{sign}(a_r) \neq 0$, so f is strictly monotone in the interval $[0, c]$ (Monotonicity Theorem). So

$$\begin{aligned} a_r > 0 \;&\Rightarrow\; a_0 = f(0) < f(a) = 0 \;\Rightarrow\; a_0 < 0, \\ a_r < 0 \;&\Rightarrow\; a_0 = f(0) > f(a) = 0 \;\Rightarrow\; a_0 > 0. \end{aligned}$$

In both cases $a_0 a_r < 0$ and the claim is established. □

Corollary 3.3. *Let* $f(\mathrm{x}) \in R[\mathrm{x}]$ *a polynomial with* m *monomials. Then* f *has at most* $2m - 1$ *roots in* R.

Proof. Consider $f(\mathrm{x})$ and $f(-\mathrm{x})$. By previous Theorem they have both at most $m-1$ strictly positive roots in R. So $f(\mathrm{x})$ has at most $2m-2$ non-zero roots and therefore at most $2m-1$ roots in R. $\square$