REAL ALGEBRAIC GEOMETRY LECTURE NOTES
(04: 29/10/2009 - BEARBEITET 03/11/2022)

SALMA KUHLMANN

Contents

1. Ordering extensions

Definition 1.1. Let L/K be a field extension and P an ordering on K.

An ordering Q of L is said to be an **extension** (*Fortsetzung*) of P if $P \subseteq Q$, or equivalently $Q \cap K = P$.

Definition 1.2. Let L/K be a field extension and P an ordering on K. We define

$$T_L(P) := \{ \sum_{i=1}^{n} p_i y_i^2 : n \in \mathbb{N},\, p_i \in P,\, y_i \in L \}.$$

Remark 1.3. Let L/K be a field extension and P an ordering on K.

Then $T_L(P)$ is the smallest preordering of L containing P.

Corollary 1.4. *Let L/K be a field extension and P an ordering on K.*

Then P has an extension to an ordering Q of L if and only if $T_L(P)$ is a proper preordering.

2. Quadratic extensions

Theorem 2.1. *Let K be a field, $a \in K$ and define $L := K(\sqrt{a})$. Then an ordering P of K extends to an ordering Q of L if and only if $a \in P$.*

Proof.

($\Rightarrow$) Assume Q is an extension of P, then $a = (\sqrt{a})^2 \in Q \cap K = P$.

($\Leftarrow$) Let $a \in P$, without loss of generality we can assume $L \neq K$ or $\sqrt{a} \notin K$. We show that $T_L(P)$ is a proper preordering (and then the thesis follows by Corollary 1.4).

If not, there is $n \in \mathbb{N}$ and there are $x_1, \ldots, x_n, y_1, \ldots, y_n \in K$, $p_1, \ldots, p_n \in P$ such that

$$-1 = \sum_{i=1}^{n} p_i(x_i + y_i\sqrt{a})^2$$
$$= \sum_{i=1}^{n} p_i(x_i^2 + ay_i^2 + 2x_iy_i\sqrt{a}).$$

On the other hand $-1 \in K$, and since every $x \in K(\sqrt{a})$ can be written in a unique way as $x = k_1 + k_2\sqrt{a}$ with $k_1, k_2 \in K$, it follows that

$$-1 = \sum_{i=1}^{n} p_i(x_i^2 + ay_i^2) \ \in P,$$

contradiction.

□

3. Odd degree field extensions

Theorem 3.1. *Let L/K be a field extension such that $[L : K]$ is finite and odd. Then every ordering of K extends to an ordering of L.*

Proof. Otherwise, let $n \in \mathbb{N}$ the minimal odd degree of a field extension for which the theorem fails.

Let L/K be a finite field extension such that $[L : K] = n$ and let P be an ordering of K not extending to an ordering of L.

Since $\mathrm{char}(K) = 0$ Primitive Element Theorem applies and there is some $\alpha \in L \setminus K$ such that

$$L \ = \ K(\alpha) \ \cong \ K[\mathrm{x}]/(f),$$

where f is the minimal polynomial of α over K. Therefore $\deg(f) = n$, $f(\alpha) = 0$ and for every $g(\mathrm{x}) \in K[\mathrm{x}]$ such that $\deg(g) < n$, we have $g(\alpha) \neq 0$.

By Corollary 1.4, $-1 \in T_L(P)$, so

$$1 + \sum_{i=1}^{s} p_i y_i^2 = 0,$$

where $\forall\, i = 1, \ldots, s \ \ p_i \in P$, $p_i \neq 0$, $y_i \in L$, $y_i \neq 0$. Write

$$y_i = g_i(\alpha),$$

where $\forall\, i = 1, \ldots, s \ \ 0 \neq g_i(\mathrm{x}) \in K[\mathrm{x}]$ and $\deg(g_i) < n$. Since

$$1 + \sum_{i=1}^{s} p_i g_i(\alpha)^2 = 0,$$

it follows that

$$1 + \sum_{i=1}^{s} p_i g_i(\mathrm{x})^2 = f(\mathrm{x})h(\mathrm{x}), \qquad \text{for some } h(\mathrm{x}) \in K[\mathrm{x}].$$

Define $d := \max\{\deg(g_i) : i = 1, \dots, s\}$. Then $d < n$ and the polynomial $f(\mathrm{x})h(\mathrm{x})$ has degree $2d$: the coefficient of x^{2d} is of the form

$$\sum_{1=1}^{r} p_i b_i^2,$$

with $p_i \in P$ and $b_i \in K$, $b_i \neq 0$, so

$$\sum_{1=1}^{r} p_i b_i^2 \ >_P \ 0.$$

Note that $\deg(h) = 2d - n < n$ (because $d < n$) and $2d - n$ is odd.

Let $h_1(\mathrm{x})$ be an irreducible factor of $h(\mathrm{x})$ of odd degree and suppose β is a root of $h_1(\mathrm{x})$. Then

$$\deg(h_1) = [K(\beta) : K] < [L : K] = n.$$

Since $h_1(\beta) = 0$, also

$$f(\beta)h(\beta) = 1 + \sum_{i=1}^{s} p_i g_i(\beta)^2 = 0.$$

Therefore $\sum_{i=1}^{s} p_i g_i(\beta)^2 = -1 \ \in T_{K(\beta)}(P)$ and by Corollary 1.4 P does not extend to an ordering of $K(\beta)$. This is in contradiction with the minimality of n. □

4. Real closed fields

Definition 4.1. (*reell abgeschloßen*) A field K is said to be **real closed** if

(1) K is real,
(2) K has no proper real algebraic extension.

Proposition 4.2. *(Artin-Schreier, 1926) Let K be a field. The following are equivalent:*

(i) K is real closed.
(ii) K has an ordering P which does not extend to any proper algebraic extension.
(iii) K is real, has no proper algebraic extension of odd degree, and

$$K \ = \ K^2 \ \cup \ -(K^2).$$

Proof. $(i) \Rightarrow (ii)$. Trivial.

$(ii) \Rightarrow (iii)$. Let P be an ordering which does not extend to any proper algebraic extension. By Theorem 3.1, it follows that K has no proper algebraic extension of odd degree.

Let $b \in P$. Then $b = a^2$ for some $a \in K$, otherwise by Theorem 2.1 P extends to an ordering of $K(\sqrt{b})$, which is a proper algebraic extension of K.

Since $K = P \cup (-P)$ and $P = \{a^2 : a \in K\}$, we get (iii).

$(iii) \Rightarrow (i)$. Note $\mathrm{char}(K) = 0$ and $\sqrt{-1} \notin K$ since K is real.

Then $K(\sqrt{-1})$ is the only proper quadratic extension of K: if $b \in K$ but $\sqrt{b} \notin K$ (i.e. b is not a square), then $b = -a^2$ for some $a \neq 0, a \in K$, and $K(\sqrt{b}) = K(\sqrt{-1}\sqrt{a^2}) = K(\sqrt{-1})$.

Claim. Every proper algebraic extension of K contains a proper quadratic subextension.

Note that if Claim is established we are done: indeed it follows that no proper extension can be real since -1 is a square in it.

Let L/K a proper algebraic extension. Without loss of generality assume that $[L : K]$ is finite and so even. By Primitive Element Theorem we can further assume that L is a Galois extension.

Let $G = \mathrm{Gal}(L/K)$, $|G| = [L : K] = 2^a m$, $a \geqslant 1$, m odd. Let S be a 2-Sylow subgroup of G (i.e. $|S| = 2^a$) and let $E := \mathrm{Fix}(S)$. By Galois correspondence we get:

$$[E : K] = [G : S] = m \qquad \text{odd.}$$

Therefore by assumption (iii) we must have $[E : K] = [G : S] = 1$, so $G = S$ is a 2-group ($|G| = 2^a$) and it has a subgroup G_1 of index 2. By Galois correspondence, defining $F_1 := \mathrm{Fix}(G_1)$ we get a quadratic subextension of L/K. $\square$