REAL ALGEBRAIC GEOMETRY LECTURE NOTES (02: 22/10/2009 - BEARBEITET 27/10/2022)

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Contents

1. The field $\mathbb{R}(\mathrm{x})$

Let us consider again the field $\mathbb{R}(\mathrm{x})$ of the rational functions on $\mathbb{R}[\mathrm{x}]$:

Example 1.1. Let $f(\mathrm{x}) = a_n\mathrm{x}^n + a_{n-1}\mathrm{x}^{n-1} + \cdots + a_1\mathrm{x} + a_0 \in \mathbb{R}[\mathrm{x}]$ and let $k \in \mathbb{N}$ the smallest index such that $a_k \neq 0$ (and therefore actually $f(\mathrm{x}) = a_n\mathrm{x}^n + \cdots + a_k\mathrm{x}^k$). We define

$$f(\mathrm{x}) > 0 \;\Leftrightarrow\; a_k > 0 \tag{1.1}$$

and then for every $f(\mathrm{x}), g(\mathrm{x}) \in \mathbb{R}[\mathrm{x}]$ with $g(\mathrm{x}) \neq 0$ we define

$$\frac{f(\mathrm{x})}{g(\mathrm{x})} \geqslant 0 \;\Leftrightarrow\; f(\mathrm{x})g(\mathrm{x}) \geqslant 0.$$

This is a total order on

$$\mathbb{R}(\mathrm{x}) = \left\{ \frac{f(\mathrm{x})}{g(\mathrm{x})} : f(\mathrm{x}), g(\mathrm{x}) \in \mathbb{R}[\mathrm{x}] \text{ and } g(\mathrm{x}) \neq 0 \right\}$$

which makes $(\mathbb{R}(\mathrm{x}), \leqslant)$ an ordered field.

Remark 1.2. By the definition above

$$f(\mathrm{x}) = \mathrm{x} - r < 0 \qquad \forall\, r \in \mathbb{R},\ r > 0.$$

Therefore the element $\mathrm{x} \in \mathbb{R}(\mathrm{x})$ is such that

$$0 < \mathrm{x} < r \quad \forall\, r \in \mathbb{R},\ r > 0.$$

We can see that there is no other ordering on $\mathbb{R}(\mathrm{x})$ which satisfies the above property:

Proposition 1.3. *Let $\leqslant$ be the ordering on $\mathbb{R}(\mathrm{x})$ defined in (1.1). Then $\leqslant$ is the unique ordering on $\mathbb{R}(\mathrm{x})$ such that*

$$0 < \mathrm{x} < r \quad \forall r \in \mathbb{R},\ r > 0.$$

Proof. Assume that $\leqslant$ is an ordering on $\mathbb{R}(\mathrm{x})$ such that

$$0 < \mathrm{x} < r \quad \forall r \in \mathbb{R},\ r > 0.$$

Then (see Proposition 2.4 of last lecture)

$$0 < \mathrm{x}^m < r \quad \forall m \geqslant 1,\ m \in \mathbb{N},\ \forall r > 0,\ r \in \mathbb{R}.$$

Let $f(\mathrm{x}) = a_n\mathrm{x}^n + a_{n-1}\mathrm{x}^{n-1} + \cdots + a_k\mathrm{x}^k \in \mathbb{R}[\mathrm{x}]$ with $k \in \mathbb{N}$ the smallest index such that $a_k \neq 0$. We want to prove that $\operatorname{sign}(f) = \operatorname{sign}(a_k)$.

Let $g(\mathrm{x}) = a_n\mathrm{x}^{n-k} + \cdots + a_{k+1}\mathrm{x} + a_k$. Then $f(\mathrm{x}) = \mathrm{x}^k g(\mathrm{x})$.

If $k = 0$, then $f(\mathrm{x}) = g(\mathrm{x})$. Otherwise $f(\mathrm{x}) \neq g(\mathrm{x})$, and since $\operatorname{sign}(f) = \operatorname{sign}(\mathrm{x}^k)\operatorname{sign}(g)$ and $\operatorname{sign}(\mathrm{x}^k) = 1$, it follows that $\operatorname{sign}(f) = \operatorname{sign}(g)$. We want $\operatorname{sign}(g) = \operatorname{sign}(a_k)$.

If $g(\mathrm{x}) = a_k$ we are done. Otherwise let $h(\mathrm{x}) = a_n\mathrm{x}^{n-k-1} + \cdots + a_{k+2}\mathrm{x} + a_{k+1}$. Then $g(\mathrm{x}) = a_k + \mathrm{x}h(\mathrm{x})$ and $h(\mathrm{x}) \neq 0$. Since $|\mathrm{x}^m| < 1$ for every $m \in \mathbb{N}$, we get

$$|h(\mathrm{x})| \leqslant |a_n| + \cdots + |a_{k+1}| := c > 0, \qquad c \in \mathbb{R}.$$

Then

$$|\mathrm{x}h(\mathrm{x})| \leqslant c|\mathrm{x}| < |a_k|,$$

otherwise $|\mathrm{x}| \geqslant \frac{|a_k|}{c}$, contradiction.

Therefore $\operatorname{sign}(g) = \operatorname{sign}(a_k + \mathrm{x}h) = \operatorname{sign}(a_k)$, as required (Note that one needs to verify that $|a| > |b| \Rightarrow \operatorname{sign}(a+b) = \operatorname{sign}(a)$).

□

We now want to classify all orderings on $\mathbb{R}(\mathrm{x})$ which make it into an ordered field. For this we need the notion of Dedekind cuts.

2. Dedekind cuts

Notation 2.1. Let $(\Gamma, \leqslant)$ be a non-empty totally ordered set and let $L, U \subseteq \Gamma$. If we write

$$L < U$$

we mean that

$$x < y \quad \forall x \in L,\ \forall y \in U.$$

(Similarly for $L \leqslant U$)

Definition 2.2. (*Dedekindschnitt*) Let $(\Gamma, \leqslant)$ be a totally ordered set. A **Dedekind cut** of $(\Gamma, \leqslant)$ is a pair (L, U) such that $L, U \subseteq \Gamma$, $L \cup U = \Gamma$ and $L < U$.

Remark 2.3. Since $L < U$ it follows that $L \cap U = \varnothing$. Therefore the subsets L, U form a partition of Γ (The letter "L" stands for "lower cut" and the letter "U" for "upper cut").

Example 2.4. Let $(\Gamma, \leqslant)$ be a non-empty totally ordered set. For every $\gamma \in \Gamma$ we can consider the following two Dedekind cuts:

$$\gamma_- := (]-\infty, \gamma[,\ [\gamma, \infty[)$$
$$\gamma_+ := (]-\infty, \gamma],\ ]\gamma, \infty[)$$

Moreover if we take $L, U \in \{\varnothing, \Gamma\}$, then we have two more cuts:

$$-\infty := (\varnothing, \Gamma), \quad +\infty := (\Gamma, \varnothing)$$

Example 2.5. Consider the Dedekind cut (L, U) of $(\mathbb{Q}, \leqslant)$ given by

$$L = \{x \in \mathbb{Q} : x < \sqrt{2}\} \quad \text{and} \quad U = \{x \in \mathbb{Q} : x > \sqrt{2}\}.$$

Then there is no $\gamma \in \mathbb{Q}$ such that $(L, U) = \gamma_-$ or $(L, U) = \gamma_+$.

Definition 2.6. (*trivialen und freie Schnitte*) Let (L, U) be a Dedekind cut of a totally ordered set $(\Gamma, \leqslant)$. If $(L, U) = \pm\infty$ or there is some $\gamma \in \Gamma$ such that $(L, U) = \gamma_+$ or $(L, U) = \gamma_-$ (as defined in 2.4), then (L, U) is said to be a **trivial** (or **realized**) Dedekind cut. Otherwise it is said to be a **free** Dedekind cut (or **gap**).

Exercise 2.7. A Dedekind cut (L, U) of a totally ordered set $(\Gamma, \leqslant)$ is free if $L \neq \varnothing$, $U \neq \varnothing$, L has no last element and U has no least element. Show that a totally ordered set $(\Gamma, \leqslant)$ is Dedekind complete if and only if $(\Gamma, \leqslant)$ has no free Dedekind cuts.

Definition 2.8. (*Dedekindvollständing*) A totally ordered set $(\Gamma, \leqslant)$ is said to be **Dedekind complete** if for every pair (L, U) of subsets of Γ with $L \neq \varnothing$, $U \neq \varnothing$ and $L \leqslant U$, there exists $\gamma \in \Gamma$ such that

$$L \leqslant \gamma \leqslant U.$$

Examples 2.9.

- The ordered set of the reals $(\mathbb{R}, \leqslant)$ is Dedekind complete, i.e. the set of Dedekind cuts of $(\mathbb{R}, \leqslant)$ is $\{a_\pm : a \in \mathbb{R}\} \cup \{-\infty, +\infty\}$.
- We have already seen in 2.5 that $(\mathbb{Q}, \leqslant)$ is not Dedekind complete. We can generalize 2.5: for every $\alpha \in \mathbb{R} - \mathbb{Q}$ we have the gap given by $(\ ]-\infty, \alpha[\ \cap \mathbb{Q},\ ]\,\alpha, \infty\,[\ \cap \mathbb{Q})$.

3. The orderings on $\mathbb{R}(\mathrm{x})$

Theorem 3.1. *There is a bijection between the set of the orderings on* $\mathbb{R}(\boldsymbol{x})$ *and the set of the Dedekind cuts of* $\mathbb{R}$.

Proof. Let $\leqslant$ be an ordering on $\mathbb{R}(\mathrm{x})$. Consider the sets $L = \{v \in \mathbb{R} : v < \mathrm{x}\}$ and $U = \{w \in \mathbb{R} : \mathrm{x} < w\}$. Then $\mathcal{C}_{\mathrm{X}}^{\leqslant} := (L, U)$ is a Dedekind cut of $\mathbb{R}$. (Note that if $\leqslant$ is the order defined in 1.1 then $\mathcal{C}_{\mathrm{X}}^{\leqslant} = 0_+$). So we can define a map

$$\{\leqslant : \leqslant \text{ is an ordering on } \mathbb{R}(\mathrm{x})\} \xrightarrow{C} \{(L,U) : (L,U) \text{ is a Dedekind cut of } \mathbb{R}\}$$

$$\leqslant \qquad \mapsto \qquad \mathcal{C}_{\mathrm{x}}^{\leqslant}$$

We now want to find a map
$\{(L,U) : (L,U) \text{ is a Dedekind cut of } \mathbb{R}\} \longrightarrow \{\leqslant : \leqslant \text{ is an ordering on } \mathbb{R}(\mathrm{x})\}$

which is the inverse of C. Every Dedekind cut of $(\mathbb{R}, \leqslant)$ is of the form $-\infty$, a_-, a_+, $+\infty$, with $a \in \mathbb{R}$. With a change of variable, respectively, $\mathrm{y} := -1/\mathrm{x}$, $\mathrm{y} := a - \mathrm{x}$, $\mathrm{y} := \mathrm{x} - a$, $\mathrm{y} := 1/\mathrm{x}$, we obtain an ordering on $\mathbb{R}(\mathrm{y})$ such that

$$0 < \mathrm{y} < r \quad \forall\, r \in \mathbb{R},\ r > 0.$$

We have seen in 1.3 that there is only one ordering with such a property, so we have a well-defined map from the set of the Dedekind cuts of $(\mathbb{R}, \leqslant)$ into the set of orderings of $\mathbb{R}(\mathrm{x})$. It is precisely the inverse of C.

□

4. Order preserving embeddings

Definition 4.1. (*ordungstreue Einbettung*) Let $(K, \leqslant)$ and $(F, \leqslant)$ be ordered fields. An injective homomorphism of fields

$$\varphi\colon K \hookrightarrow F$$

is said to be an **order preserving embedding** if

$$a \leqslant b \;\Rightarrow\; \varphi(a) \leqslant \varphi(b) \qquad \forall\, a, b \in K.$$

Theorem 4.2 (Hölder)**.** *Let $(K, \leqslant)$ be an Archimedean ordered field. Then there is an order preserving embedding*

$$\varphi\colon K \hookrightarrow \mathbb{R}.$$

Proof. Let $a \in K$. Consider the sets

$$I_a := \,]-\infty, a]_K \,\cap\, \mathbb{Q} \quad \text{and} \quad F_a := [a, \infty[_K \cap\, \mathbb{Q}.$$

Then $I_a \leqslant F_a$ and $I_a \cup F_a = \mathbb{Q}$. So we can define

$$\varphi(a) := \sup I_a = \inf F_a \in \mathbb{R}.$$

Since K is Archimedean, φ is well-defined. Note that $\varphi(a) \in \mathbb{R}$ and

$$I_a + I_b = \{x + y : x \in I_a,\ y \in I_b\} \subseteq I_{a+b}$$

and

$$F_a + F_b \subseteq F_{a+b},$$

then $\varphi(a) + \varphi(b) \leqslant \varphi(a+b)$ and $\varphi(a) + \varphi(b) \geqslant \varphi(a+b)$. This proves that φ is additive. Similarly one gets $\varphi(ab) = \varphi(a)\varphi(b)$.

□