REAL ALGEBRAIC GEOMETRY LECTURE NOTES (01: 20/10/2009 - BEARBEITET 25/10/2022)

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Contents

Convention: When a new definition is given, the German name appears between brackets.

1. Orderings

Definition 1.1. (*partielle Anordnung*) Let Γ be a non-empty set and let $\leqslant$ be a relation on Γ such that:

(i) $\gamma \leqslant \gamma \quad \forall \gamma \in \Gamma$,

(ii) $\gamma_1 \leqslant \gamma_2, \gamma_2 \leqslant \gamma_1 \;\Rightarrow\; \gamma_1 = \gamma_2 \quad \forall \gamma_1, \gamma_2 \in \Gamma$,

(iii) $\gamma_1 \leqslant \gamma_2,\; \gamma_2 \leqslant \gamma_3 \;\Rightarrow \gamma_1 \leqslant \gamma_3 \quad \forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma$.

Then $\leqslant$ is a **partial order** on Γ and $(\Gamma, \leqslant)$ is said to be a **partially ordered set**.

Example 1.2. Let X be a non-empty set. For every $A, B \subseteq X$, the relation

$$A \leqslant B \iff A \subseteq B,$$

is a partial order on the power set $\mathcal{P}(X) = \{A : A \subseteq X\}$.

Definition 1.3. (*totale Anordung*) A partial order $\leqslant$ on a set Γ is said to be **total** if

$$\forall \gamma_1, \gamma_2 \in \Gamma \quad \gamma_1 \leqslant \gamma_2 \text{ or } \gamma_2 \leqslant \gamma_1.$$

Notation 1.4. If $(\Gamma, \leqslant)$ is a partially ordered set and $\gamma_1, \gamma_2 \in \Gamma$, then we write:

$$\begin{aligned} \gamma_1 < \gamma_2 &\Leftrightarrow \gamma_1 \leqslant \gamma_2 \text{ and } \gamma_1 \neq \gamma_2, \\ \gamma_1 \geqslant \gamma_2 &\Leftrightarrow \gamma_2 \leqslant \gamma_1, \\ \gamma_1 > \gamma_2 &\Leftrightarrow \gamma_2 \leqslant \gamma_1 \text{ and } \gamma_1 \neq \gamma_2. \end{aligned}$$

Examples 1.5. Let $\Gamma = \mathbb{R} \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}\}$.

(1) For every $(a_1, b_1), (a_2, b_2) \in \mathbb{R} \times \mathbb{R}$ we can define
$$(a_1, b_1) \leqslant (a_2, b_2) \iff a_1 \leqslant a_2 \text{ and } b_1 \leqslant b_2.$$
Then $(\mathbb{R} \times \mathbb{R}, \leqslant)$ is a partially ordered set.

(2) For every $(a_1, b_1), (a_2, b_2) \in \mathbb{R} \times \mathbb{R}$ we can define
$$(a_1, b_1) \leqslant_l (a_2, b_2) \iff [a_1 < a_2] \text{ or } [a_1 = a_2 \text{ and } b_1 \leqslant b_2].$$
Then $(\mathbb{R} \times \mathbb{R}, \leqslant_l)$ is a totally ordered set. (Remark: the "l" stands for "lexicographic").

2. Ordered fields

Definition 2.1. (*angeordneter Körper*) Let K be a field. Let $\leqslant$ be a total order on K such that:

(i) $x \leqslant y \Rightarrow x + z \leqslant y + z \quad \forall x, y, z \in K$,

(ii) $0 \leqslant x,\ 0 \leqslant y \Rightarrow 0 \leqslant xy \quad \forall x, y \in K$.

Then the pair $(K, \leqslant)$ is said to be an **ordered field**.

Examples 2.2. The field of the rational numbers $(\mathbb{Q}, \leqslant)$ and the field of the real numbers $(\mathbb{R}, \leqslant)$ are ordered fields, where $\leqslant$ denotes the usual order.

Definition 2.3. (*formal reell Körper*) A field K is said to be **(formally) real** if there is an order $\leqslant$ on K such that $(K, \leqslant)$ is an ordered field.

Proposition 2.4. *Let $(K, \leqslant)$ be an ordered field. The following hold:*

- $a \leqslant b \Leftrightarrow 0 \leqslant b - a \quad \forall a, b \in K$
- $0 \leqslant a^2 \quad \forall a \in K$
- $a \leqslant b,\ 0 \leqslant c \Rightarrow ac \leqslant bc \quad \forall a, b, c \in K$
- $0 < a \leqslant b \Rightarrow 0 < 1/b \leqslant 1/a \quad \forall a, b \in K$
- $0 < n \quad \forall n \in \mathbb{N}$

Remark 2.5. If K is a real field then $\operatorname{char}(K) = 0$ and K contains a copy of $\mathbb{Q}$.

Notation 2.6. Let $(K, \leqslant)$ be an ordered field and let $a \in K$.
$$\operatorname{sign}(a) := \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

$$|a| := \operatorname{sign}(a)a.$$

Fact 2.7. Let $(K, \leqslant)$ be an ordered field and let $a, b \in K$. Then

(*i*) $\operatorname{sign}(ab) = \operatorname{sign}(a)\operatorname{sign}(b)$,

(*ii*) $|ab| = |a||b|$,

(*iii*) $|a+b| \leqslant |a| + |b|$.

3. Archimedean fields

Definition 3.1. (*archimedischer Körper*) Let $(K, \leqslant)$ be an ordered field. We say that K is **Archimedean** if

$$\forall\, a \in K \;\; \exists n \in \mathbb{N} \text{ such that } \; a < n.$$

Definition 3.2. Let $(\Gamma \leqslant)$ be an ordered set and let $\Delta \subseteq \Gamma$. Then

- Δ is **cofinal** (*kofinal*) in Γ if
$$\forall\, \gamma \in \Gamma \;\; \exists \delta \in \Delta \text{ such that } \; \gamma \leqslant \delta.$$
- Δ is **coinitial** (*koinitial*) in Γ if
$$\forall\, \gamma \in \Gamma \;\; \exists \delta \in \Delta \text{ such that } \; \delta \leqslant \gamma.$$
- Δ is **coterminal** (*koterminal*) in Γ if Δ is cofinal and coinitial in Γ.

Example 3.3. Let $(K \leqslant)$ be an Archimedean field. Then $\mathbb{N}$ is cofinal in K, $-\mathbb{N}$ is coinitial in K and $\mathbb{Z} = -\mathbb{N} \cup \mathbb{N}$ is coterminal in K.

Remark 3.4.

- If $(K, \leqslant)$ is an Archimedean field and $Q \subseteq K$ is a subfield, then $(Q, \leqslant)$ is an Archimedean field.
- $(\mathbb{R}, \leqslant)$ is an Archimedean field and therefore also $(\mathbb{Q}, \leqslant)$ is.

Remark 3.5. Let $(K, \leqslant)$ be an ordered field. Then K is Archimedean if and only if $\forall\, a, b \in K^* \; \exists n \in \mathbb{N}$ such that

$$|a| \leqslant n|b| \text{ and } |b| \leqslant n|a|.$$

Example 3.6. Let $\mathbb{R}[\mathrm{x}]$ be the ring of the polynomials with coefficients in $\mathbb{R}$. We denote by $ff(\mathbb{R}[\mathrm{x}])$ the field of the rational functions of $\mathbb{R}[\mathrm{x}]$, i.e.

$$ff(\mathbb{R}[\mathrm{x}]) = \mathbb{R}(\mathrm{x}) := \left\{ \frac{f(\mathrm{x})}{g(\mathrm{x})} : f(\mathrm{x}), g(\mathrm{x}) \in \mathbb{R}[\mathrm{x}] \text{ and } g(\mathrm{x}) \neq 0 \right\}.$$

Let $f(\mathrm{x}) = a_n\mathrm{x}^n + a_{n-1}\mathrm{x}^{n-1} + \cdots + a_1\mathrm{x} + a_0 \in \mathbb{R}[\mathrm{x}]$ and let $k \in \mathbb{N}$ the smallest index such that $a_k \neq 0$ (and therefore actually $f(\mathrm{x}) = a_n\mathrm{x}^n + \cdots + a_k\mathrm{x}^k$). We define

$$f(\mathrm{x}) > 0 \;\Leftrightarrow\; a_k > 0$$

and then for every $f(\mathrm{x}), g(\mathrm{x}) \in \mathbb{R}[\mathrm{x}]$ with $g(\mathrm{x}) \neq 0$ we define

$$\frac{f(\mathrm{x})}{g(\mathrm{x})} \geqslant 0 \;\Leftrightarrow\; f(\mathrm{x})g(\mathrm{x}) \geqslant 0.$$

This is a total order on $K = ff(\mathbb{R}[\mathrm{x}])$ which makes $(K, \leqslant)$ an ordered field. We claim that $(K, \leqslant)$ contains

(*i*) an infinite positive element, i.e.

$$\exists\, A \in K \text{ such that } A > n \;\; \forall\, n \in \mathbb{N},$$

(*ii*) an infinitesimal positive element, i.e.

$$\exists\, a \in K \text{ such that } 0 < a < 1/n \;\; \forall\, n \in \mathbb{N}.$$

For instance the element $\mathrm{x} \in K$ is infinitesimal and the element $1/\mathrm{x} \in K$ is infinite. Therefore $(K, \leqslant)$ is not Archimedean.