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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 6 - Solution

1. (a) Consider an element $\alpha > 0$ of K and the polynomial $f(x) = x^2 - \alpha$. Then for instance $f(0) = -\alpha < 0$ and $f(1+\alpha) = 1+\alpha+\alpha^2 > 0$. So there exists $c \in]0, 1+\alpha[$ such that $f(c) = c^2 - \alpha = 0$. Thus $c = \sqrt{\alpha} \in K$.

   (b) Consider a polynomial $f(x) = d_n x^n + d_{n-1}x^{n-1} + \cdots + d_0 \in K[x]$ with n odd. We suppose without loss of generality that $d_n > 0$. For $|x| > 1$, we have
   $$|d_{n-1}x^{n-1} + \cdots + d_0| < (|d_{n-1}| + \cdots + |d_0|)|x|^{n-1}.$$
   Then take $b > 1$ such that $b > \dfrac{|d_{n-1}| + \cdots + |d_0|}{d_n}$. We get that
   $$d_n b^n > (|d_{n-1}| + \cdots + |d_0|)b^{n-1} > |d_{n-1}b^{n-1} + \cdots + d_1 b + d_0| \geq -(d_{n-1}b^{n-1} + \cdots + d_0)$$
   which means that $f(b) > 0$.
   Moreover, we get that
   $$d_n(-b)^n = -d_n b^n < -(|d_{n-1}| + \cdots + |d_0|)b^{n-1} < -|d_{n-1}b^{n-1}| - \cdots - |d_1 b| - |d_0| \leq -d_{n-1}(-b)^{n-1} - \cdots + d_1 b - d_0$$
   which means that $f(-b) < 0$.
   So $f(-b) < 0 < f(b)$, which implies that there exists $c \in]-b, b[$ such that $f(c) = 0$.
   Then apply the second Corollary of Lecture of 03-11-09 to conclude that K is real closed.

2. Let $(R, \geq)$ be a real closed field. Consider
   $$Pos(R) := \{x \in R \mid x > 0\}.$$
   (a) We have:
   - $Pos(R)$ is stable by multiplication: $g_1 > 0, g_2 > 0 \Rightarrow g_1.g_2 > 0$;
   - $Pos(R)$ is stable by taking the inverse: $g > 0 \Rightarrow \dfrac{1}{g} > 0$;
   - $Pos(R)$ is a subgroup of R;
   - $Pos(R)$ is an ordered set by retriction of the ordering on R;
   - $Pos(R)$ is an ordered subgroup of R: $g_1 \geq g_2 > 0$ and $h > 0 \Rightarrow g_1.h = h.g_1 \geq$

$g_2.h = h.g_2 > 0$.

(b) Take any element $a > 0$ of $Pos(R)$. For any $n \in \mathbb{N}^*$, the polynomial $f_n(x) = x^n - a$ is such that

$f(0) = -a < 0 < f(1+a) = a^n + na^{n-1} + \cdots + na + 1 - a = a^n + na^{n-1} + \cdots + (n-1)a + 1.$

By the intermediate value theorem, there exists $c \in]0, 1+a[$ such that $f(c) = 0$. Thus $c = \sqrt[n]{a} > 0$ exists in R, which means that $\exists \sqrt[n]{a} \in Pos(R)$ for any $n \in \mathbb{N}$.

3. We consider the **Motzkin polynomial**

$$m(X,Y) = 1 - 3X^2Y^2 + X^2Y^4 + X^4Y^2.$$

(a) Take $a = 1$, $b = X^2Y^4$ and $c = X^4Y^2$. We have $a + b + c = 1 + X^2Y^4 + X^4Y^2 \leq 3\sqrt[3]{abc} = 3\sqrt[3]{X^6Y^6} = 3(XY)^2$. Thus

$$m(X,Y) = a + b + c - 3X^2Y^2 \geq 3(XY)^2 - 3(XY)^2 = 0.$$

(b) We consider a polynomial $f = f_1^2 + \cdots + f_k^2$ for some $f_i(\underline{X}) \in \mathbb{R}[\underline{X}]$ with $f_1 \neq 0$. Since $f_1 \neq 0$, there exists $(\underline{x}) \in \mathbb{R}^n$ such that $f_1(\underline{x}) \neq 0$. Equivalently $f_1(\underline{x})^2 > 0$. But for any i, $f_i(\underline{x})^2 \geq 0$. So

$$f(\underline{x}) = f_1(\underline{x})^2 + \cdots + f_k(\underline{x})^2 > 0,$$

which implies that $f \neq 0$.

Write $d := \max\{\deg(f_i),\ i = 1, \ldots, k\}$, and $f_{i_0}, \ldots, f_{i_l}$ all the polynomials that have such degree d (note that $l \leq k$). For any $j = 1, \ldots, l$, $f_{i_j} = g_{i_j} + h_{i_j}$ where g_{i_j} is non zero homogenous of degree d and h_{i_j} has degree $< d$. We get that $f_{i_j}^2 = g_{i_j}^2 + 2g_{i_j}h_{i_j} + h_{i_j}^2$ where $g_{i_j}^2$ is > 0 homogenous of degree $2d$, and $2g_{i_j}h_{i_j} + h_{i_j}^2$ has degree $< 2d$. For the others polynomials f_i, we note that f_i^2 has degree $< 2d$. So $f = f_1^2 + \cdots f_k^2$ can be written as $f = g_{i_0}^2 + \cdots + g_{i_l}^2 + h$ with $g_{i_0}^2 + \cdots + g_{i_l}^2 > 0$ homogenous of degree $2d$ and h of degree $< 2d$. Therefore f has degree $2d$.

(c) We suppose that the Motzkin polynomial can be written $m = f_1^2 + \cdots f_k^2$ for some $f_i(X,Y) \in \mathbb{R}[X,Y]$. Since $\deg(m) = 6$, we must have $\max\{\deg(f_i),\ i = 1, \ldots, k\} = 3$. A base of the vector space of polynomials of degree ≤ 3 is given by

$$\{1, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3\}.$$

If X^3, X^2, X respectively, appears in some f_i, then X^6, X^4, X^2 respectively, would appear in m with positive coefficient, which is not the case. So X^3, X^2, X do not appear in m. With the same argument, Y^3, Y^2, Y do not appear either.

(d) We write $f_i(X,Y) = a_i + b_iXY + c_iX^2Y + d_iXY^2$ for any $i = 1, \ldots, k$. Then we would have $f_i(X,Y)^2 = b_i^2X^2Y^2 + \textit{other terms}$, which means that

$$f(X,Y) = (\sum i = 1^k b_i^2)X^2Y^2 + \textit{other terms}.$$

Identifying the terms with same degree, we would have

$$\sum_{i=1}^{k} b_i^2 = -3$$

which is clearly false in $\mathbb{R}$. Contradiction.

4. Denote $\underline{X} = (X_1, \ldots, X_n)$ for some fixed $n \in \mathbb{N}^*$ and let $d \in \mathbb{N}$. Consider some non zero polynomial $f(\underline{X}) \in \mathbb{R}[\underline{X}]$ of total degree less than or equal to d.

(a) Consider an arbitrary monic monomial of $f(\underline{X})$, say $\underline{X}^{\underline{i}} = X_1^{i_1} \cdots X_n^{i_n}$ for some multi-index $\underline{i} = (i_1, \ldots, i_n)$, such that its degree $i_1 + \cdots + i_n \leq d$. Then substituting the variables X_i by $\dfrac{X_i}{X_0}$ for all $i = 1, \ldots, n$ (it is a well-defined change of variable, we use same notations for simplicity), we get that

$$\underline{X}^{\underline{i}} \leftrightarrow \frac{X_1^{i_1} \cdots X_n^{i_n}}{X_0^{i_1+\cdots+i_n}}$$

and so

$$X_0^d \underline{X}^{\underline{i}} \leftrightarrow X_0^{d-(i_1+\cdots+i_n)} X_1^{i_1} \cdots X_n^{i_n}$$

This new monomial has degree $d - (i_1 + \cdots + i_n) + i_1 + \cdots + i_n = d$ for all i. We extend this procedure to all the monomials of $f(x)$ since the change of variable is done termwise and by distributivity of the multiplication by X_0^d. We obtain a homogenous polynomial $\overline{f}(X_0, X_1, \ldots, X_n)$ of degree d.

(b) Denote by $h : V_{d,n} \to F_{d,n+1}$ the homogenization map

$$h : f(\underline{X}) \mapsto \overline{f}(X_0, X_1, \ldots, X_n).$$

It is a linear map: for any $\alpha, \beta \in \mathbb{R}$, for any $f(\underline{X}), g(\underline{X}) \in V_{d,n}$, $h(\alpha f + \beta g) = \alpha h(f) + \beta h(g) = \alpha \overline{f}(X_0, X_1, \ldots, X_n) + \beta \overline{g}(X_0, X_1, \ldots, X_n)$ (follows again from the fact that the change of variable is done termwise, concerning only the monic monomial regardless of the coefficient, and by distributivity of the multiplication by X_0^d).

The compositional inverse h^{-1} of h is given by:

$$\begin{array}{llll} h : & V_{d,n} & \to & F_{d,n+1} \\ & \overline{f}(X_0, X_1, \ldots, X_n) & \mapsto & h(\overline{f})(\underline{X}) := \overline{f}(1, X_1, \ldots, X_n). \end{array}$$

and is also clearly linear.

(c) Let d be an even number. We show that $f \geq 0$ on $\mathbb{R}^n$ implies $\overline{f} \geq 0$ on $\mathbb{R}^{n+1}$. For any $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$, note that by definition, if $x_0 \neq 0$, $\overline{f}(x_0, x_1, \ldots, x_n) = x_0^d f\left(\dfrac{x_1}{x_0}, \ldots, \dfrac{x_n}{x_0}\right)$. By hypothesis, $f\left(\dfrac{x_1}{x_0}, \ldots, \dfrac{x_n}{x_0}\right) \geq 0$ and, since d is even, $x_0^d > 0$. If $x_0 = 0$, since a polynomial is a continuous map, we have $\overline{f}(0, x_1, \ldots, x_n) = \lim_{\epsilon \to 0} \overline{f}(\epsilon, x_1, \ldots, x_n)$. But for any $\epsilon \in \mathbb{R}^*$, we just showed that $\overline{f}(\epsilon, x_1, \ldots, x_n) \geq 0$. So $\overline{f}(0, x_1, \ldots, x_n) \geq 0$.

To show that $\overline{f} \geq 0$ on $\mathbb{R}^{n+1}$ implies $f \geq 0$ on $\mathbb{R}^n$, consider that for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$, we have

$$f(x_1, \ldots, x_n) = \overline{f}(1, x_1, \ldots, x_n) \geq 0$$

by hypothesis.

(d) Suppose that $f = \sum_{i=1}^{k} f_i^2$ for some non zero f_i's in $\mathbb{R}[\underline{X}]$. First, remark that $deg(f_i) \leq d/2$ since $deg(f) \leq d$ (see preceding exercise). We have $\overline{f}(X_0,X_1,\ldots,X_n) = \sum_{i=1}^{k} \left[X_0^{d/2} f_i \left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0} \right) \right]^2$. But this is a sum of squares of forms of degree $d/2$.

Suppose now that $\overline{f}(X_0,X_1,\ldots,X_n) = \sum_{i=1}^{k} g_i(X_0,\ldots,X_n)^2$. Then consider $f(\underline{X}) = \overline{f}(1,X_1,\ldots,X_n) = \sum_{i=1}^{k} g_i(1,X_1,\ldots,X_n)^2$, and put $f_i(\underline{X}) := g_i(1,X_1,\ldots,X_n)$.

(e)

Theorem 0.1 *For d even, $\tilde{P}_{d,n} = \tilde{\sum}_{d,n}$ if and only if $n = 1$ (polynomials in one variable of any degree), or $d = 2$ (polynomials in any number of variables of degree 2), or ($n = 2$ and $d = 2$ or 4) (polynomials in 2 variables of degree 2 or 4).*

(f) The polynomial $\overline{m}$ is the homogenization of m, the Motzkin polynomial of exercise 3. For this one we proved in 3. that it is PSD but not sum of squares. Then use the equivalences proved in 4.(c) and 4.(d) to show also that $\overline{m}$ is PSD but not sum of squares.