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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 14

These exercises will be collected Tuesday 9 February in the mailbox number 15 of the Mathematics department.

1. Consider a system of valued $\mathbb{R}$-vector spaces $S = [\mathbb{Q},\{B(q); q \in \mathbb{Q}\}]$ with $B(q) \simeq \mathbb{R}$ for any $q \in \mathbb{Q}$, the corresponding Hahn sum $\coprod_{\gamma\in\Gamma} B(\gamma)$ and an automorphism $\sigma : \mathbb{Q} \to \mathbb{Q}$ of the ordered set $(\mathbb{Q}, \leq)$. Then show that the map
$$\tilde{\sigma} : \coprod_{\gamma\in\Gamma} B(\gamma) \to \coprod_{\gamma\in\Gamma} B(\gamma)$$
such that $\tilde{\sigma}(s)(q) := s(\sigma(q))$ for any $s \in \coprod_{\gamma\in\Gamma} B(\gamma)$ and any $q \in \mathbb{Q}$, is an automorphism of valued vector spaces.

2. **Definition 0.1** *Let $(G, +, \leq)$ be an ordered abelian group. A subgroup $C \subset G$ is said to be convex if for any $c_1,c_2 \in C$ and for any $x \in G$ such that $c_1 \leq x \leq c_2$, then $x \in C$.*

   (a) Let $(G, +, \leq)$ be an ordered abelian group. Show that the set of all convex subgroups of G is totally ordered by $\subseteq$, contains the trivial subgroup $\{0\} \subset G$ and G, and is stable under union and intersection.

   (b) Given a convex subgroup C of an ordered abelian group $(G, +, \leq)$, we define on the group $(G/C,+)$ a induced relation $\leq$ by
$$\text{for any } x,y \in G,\ x \leq y \Rightarrow x + C \leq y + C.$$
   Show that $(G/C, +, \leq)$ is an ordered abelian group.

Definition 0.2 *Given two ordered groups $(G_1, \leq)$ and $(G_2, \leq)$, and a group morphism $h : (G_1, \leq) \to (G_2, \leq)$, we say that h is an order preserving morphism if for any $x,y \in G_1$, $x \leq y \Rightarrow h(x) \leq h(y)$.*

(c) With the same hypothesis as in Question (b), prove that the canonical projection $\Pi : G \to G/C$ is an order preserving morphism.

3. **Definition 0.3** • *A sequence $s := (a_\rho)_{\rho\in\Lambda}$ (Λ being a well-ordered set) in a valued vector space (V,v) is said to be* ***pseudo-Cauchy*** *if for any $\rho < \sigma < \tau$, we have $v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma)$.*
   • *For any $\rho \in \Lambda$, we define $\gamma_\rho := v(a_{\rho+1} - a_\rho)$. Then the sequence $(\gamma_\rho)_{\rho\in\Lambda}$ is strictly increasing in Γ.*
   • *If there exists $\rho_0 \in \Lambda$ such that for any $\rho \geq \rho_0$, $v(a_\rho) = v(a_{\rho_0})$, then we define this value to be the* ***ultimate value*** *of s: $Ult(s) := v(a_{\rho_0})$.*
   • *An element $x \in V$ is said to be a* ***pseudo-limit*** *of a pseudo-Cauchy sequence $s := (a_\rho)_{\rho\in\Lambda}$ if $v(x - a_\rho) = \gamma_\rho$ for any $\rho \in \Lambda$.*
   • *The* ***breadth*** *of a pseudo-Cauchy sequence $s := (a_\rho)_{\rho\in\Lambda}$ is by definition $Br(s) := \{y \in V \mid v(y) > \gamma_\rho \ \forall\rho\}$.*

   Consider the ordered set $\Gamma = \mathbb{N}.\mathbb{N}$ which has order type ω^2 (i.e. the set $\mathbb{N} \times \mathbb{N}$ endowed with the lexicographic order: see ÜA Blatt 13). Consider the system of ordered $\mathbb{Q}$-vector spaces $S := [\Gamma, \{B(\gamma); \gamma \in \Gamma\}]$ where $B(\gamma) = \mathbb{R}$ for any γ, and the corresponding Hahn sum $M := \coprod_{\gamma\in\Gamma} B(\gamma)$ and Hahn product $N := \mathbf{H}_{\gamma\in\Gamma} B(\gamma)$ endowed as usual with the valuation $v := v_{\min}$.
   We define the following sequences in M:

   - $s^{(1)} := (a_n^{(1)})_{n\in\mathbb{N}^*}$
     where for any $(k,l) \in \Gamma$, $a_n^{(1)}(k,l) := \left| \begin{array}{lll} l^k & \text{if} & k \leq n,\ l \leq n \\ 0 & \text{if not} & \end{array} \right.$
   - $s^{(2)} := (a_n^{(2)})_{n\in\mathbb{N}^*}$
     where for any $(k,l) \in \Gamma$, $a_n^{(2)}(k,l) := \left| \begin{array}{lll} n^k & \text{if} & k \leq n,\ l = n \\ 0 & \text{if not} & \end{array} \right.$
   - $s^{(3)} := (a_n^{(3)})_{n\in\mathbb{N}^*}$
     where for any $(k,l) \in \Gamma$, $a_n^{(3)}(k,l) := \left| \begin{array}{lll} n^n & \text{if} & k = l = n \\ 0 & \text{if not} & \end{array} \right.$

   (a) Prove that these sequences are pseudo-Cauchy and compute the corresponding $\gamma_n^{(i)} := v(a_{n+1}^{(i)} - a_n^{(i)})$, $i = 1,2,3$.

   (b) Compute the correponding $Ult(s^{(i)})$ (whenever it is defined) and $Br(s^{(i)})$, $i = 1,2,3$ in M as well as in N.

   (c) For each of these 3 sequences, describe the set of all pseudo-limits in M and in N.