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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 13 - Solution

1. **Definition 0.1** *Let* (M,v) *be a* ***valued (=bewerte) module*** *and* $\Gamma = v(M \setminus \{0\})$ *its value set. Für any* $\gamma \in \Gamma$*, we define* $M^\gamma := \{x \in M \mid v(x) \geq \gamma\}$*,* $M_\gamma := \{x \in M \mid v(x) > \gamma\}$ *which are submodules of M, and* $B(M,\gamma) := M^\gamma/M_\gamma$ *which is also a module.*
   The system of modules $S(M) := [\Gamma,\{B(M,\gamma),\ \gamma \in \Gamma\}]$ *is called the* ***skeleton*** *of* (M,v)*.*
   The aim of this exercise is to prove the following lemma:
   Lemma 0.2 *The skeleton is an isomorphism invariant, i.e. if two valued modules* (M_1,v_1)*,* (M_2,v_2) *are isomorphic, then so are the corresponding two skeletons* $S(M_1)$*,* $S(M_2)$*.*

   (a) Let $h : (M_1,v_1) \to (M_2,v_2)$ be an isomorphism of valued modules, and $\Gamma_1 := v_1(M_1 \setminus \{0\})$ and $\Gamma_2 := v_2(M_2 \setminus \{0\})$ the corresponding value sets. Consider the map
   $$\tilde{h} : \Gamma_1 \to \Gamma_2,\ \ \tilde{h}(v_1(x)) := v_2(h(x)).$$
   Take $x,y \in M_1$, with $v_1(x) < v_1(y)$. Since h is an isomorphism of valued modules, this implies that $v_2(h(x)) < v_2(h(y))$, so $\tilde{h}(v_1(x)) < \tilde{h}(v_1(y))$. Thus, $\tilde{h}$ is order preserving, which implies that it is injective.
   Now, consider $v_2(z) \in \Gamma_2$. Since h is an isomorphism of valued modules, there exists a unique $x = h^{-1}(z) \in \Gamma_1$, such that $\tilde{h}^{-1}(v_2(z)) := v_1(x)$. So $\tilde{h}$ is surjective, and therefore is an isomorphism of ordered sets.
   Note that by definition h preserves the valuation (ist bewertungserhaltend).

   (b) For any $\gamma \in \Gamma_1$, we define
   $$\begin{array}{cccc} h_\gamma : & B(M_1,\gamma) & \to & B(M_2,\tilde{h}(\gamma)) \\ & \Pi^{M_1}(\gamma,x) & \mapsto & \Pi^{M_2}(\tilde{h}(\gamma),h(x)). \end{array}$$

Consider $x \in M_{1,\gamma}$ so that $\Pi^{M_1}(\gamma,x) = x + M_1^\gamma \in B(M_1,\gamma)$. Thus $v_1(x) \geq \gamma$. This implies by the preceding question that $v_2(h(x)) \geq \tilde{h}(\gamma)$. So we can define uniquely $h_\gamma(x)$ as $h(x) + M_{2,\tilde{h}(\gamma)} = \Pi^{M_2}(\tilde{h}(\gamma),h(x)) \in B(M_2,\tilde{h}(\gamma))$. The map h_γ is therefore well-defined.

Moreover, take any $z \in M_2^{\tilde{h}(\gamma)}$ so that $z + M_{2,\tilde{h}(\gamma)} = \Pi^{M_2}(\tilde{h}(\gamma),z) \in B(M_2,\tilde{h}(\gamma))$. Since $v_2(z) \geq \tilde{h}(\gamma)$, we have $v_1(h^{-1}(z)) \geq \gamma$. Then we define uniquely the inverse h_γ^{-1} of h_γ as $h_\gamma^{-1}(\Pi^{M_2}(\tilde{h}(\gamma),z)) := h^{-1}(z) + M_1^\gamma = \Pi^{M_1}(\gamma,h^{-1}(x)) \in B(M_1,\gamma)$. Thus h_γ is bijective.

Now consider $a,b \in \mathcal{Z}$ (suppose for simplicity $\mathcal{Z}$ is a ring and M_1, M_2 are $\mathcal{Z}$-modules) and $x,y \in M_1$ with $v_1(x) \geq \gamma$ and $v_1(y) \geq \gamma$. So $ax + by \in M_1$ with $v_1(ax + by) \geq \gamma$ (ultrametric triangular inequality). We have

$$\begin{aligned} h_\gamma(a\Pi^{M_1}(\gamma,x) + b\Pi^{M_1}(\gamma,y)) &= h_\gamma(\Pi^{M_1}(\gamma,ax+by)) \\ &= \Pi^{M_2}(\gamma,x)(\tilde{h}(\gamma),h(ax+by)) \\ &= \Pi^{M_2}(\gamma,x)(\tilde{h}(\gamma),ah(x)+bh(y)) \\ &= a\Pi^{M_2}(\gamma,x)(\tilde{h}(\gamma),h(x)) + b\Pi^{M_2}(\gamma,x)(\tilde{h}(\gamma),h(y)) \\ &= ah_\gamma(\Pi^{M_1}(\gamma,x)) + bh_\gamma(\Pi^{M_1}(\gamma,y)). \end{aligned}$$

For any $\gamma \in \Gamma_1$, the map h_γ is an isomorphism of modules.

2. **Definition 0.3** *Consider a* ***system of torsion free modules*** $S = [\Gamma,\{B(\gamma);\gamma \in \Gamma\}]$, *and denote by* $\prod_{\gamma\in\Gamma} B(\gamma)$ *the corresponding product module. Denote by* $\bigoplus_{\gamma\in\Gamma} B(\gamma)$ *the submodule of maps* $s \in \prod_{\gamma\in\Gamma} B(\gamma)$ *with finite support, and* $\coprod_{\gamma\in\Gamma} B(\gamma)$ *the* ***Hahn sum*** *of* S*, i.e. the valued module* $(\bigoplus_{\gamma\in\Gamma} B(\gamma),v_{\min})$ *where* $v_{\min}(s) := \min(\text{support } s)$ *for all* $s \in \bigoplus_{\gamma\in\Gamma} B(\gamma) \setminus \{0\}$.

   Denote by $\mathbf{H}_{\gamma\in\Gamma}B(\gamma)$ *the* ***Hahn product*** *of* S*, i.e. the submodule of maps* $s \in \prod_{\gamma\in\Gamma} B(\gamma)$ *with well-ordered support, also equipped with the valuation* $v_{\min}$.

   (a) Let Z be the coefficient ring of the modules $B(\gamma)$. Check that the Hahn sum and the Hahn product equipped with $v_{\min}$ are valued $\mathcal{Z}$-modules. We sketch the case of the Hahn product. Firstly, note that the linear combination of two maps s_1 and s_2 in $\prod_{\gamma\in\Gamma} B(\gamma)$ with well-ordered supports, has itself well-ordered support (indeed, the support of the linear combination is included into the union of the supports of s_1 and s_2). So $\mathbf{H}_{\gamma\in\Gamma}B(\gamma)$ is a module. Secondly, show that $v_{\min}$ is a valuation, checking the definition of a valuation:
   • $v(s) = \infty$ if and only if $s = 0$: indeed, whenever $s \neq 0$, it has a non empty well-ordered support which has a minimum, and therefore $v_{\min}(s) \neq \infty$;
   • $v(rs) = v(s)$ for any $r \in \mathcal{Z} \setminus \{0\}$: the multiplication by a scalar does not change the minimum of the support;

• by definition $v(s_1-s_2)=\min(\text{support } s_1-s_2)$. But support $s_1-s_2\subseteq$ support $s_1\cup$ support s_2. So $\min\{\text{support } s_1-s_2\}\geq\min\{\min(\text{support } s_1),\min(\text{support } s_2\}=\min\{v_{\min}(s_1),v_{\min}(s_2)\}$.
For the Hahn sum note that, since the linear combination of two maps s_1 and s_2 in $\prod_{\gamma\in\Gamma} B(\gamma)$ with finite supports, has itself finite support, the Hahn sum is a valued submodule of the Hahn product.

(b) Denote $M:=\coprod_{\gamma\in\Gamma} B(\gamma)$ and $N:=\mathbf{H}_{\gamma\in\Gamma}B(\gamma)$. Clearly, we have $v_{\min}(M\setminus\{0\})=v_{\min}(N\setminus\{0\})=\Gamma$.
Moreover:

$$\begin{aligned}
M^{\gamma} &= \{s\in M \text{ such that } \min(\text{support } s)\geq\gamma\}\\
&= \{s\in M \text{ such that } s\in\coprod_{\delta\geq\gamma} B(\delta)\};\\
M_{\gamma} &= \{s\in M \text{ such that } \min(\text{support } s)>\gamma\}\\
&= \{s\in M \text{ such that } s\in\coprod_{\delta>\gamma} B(\delta)\};\\
N^{\gamma} &= \{s\in N \text{ such that } \min(\text{support } s)\geq\gamma\}\\
&= \{s\in M \text{ such that } s\in\mathbf{H}_{\delta\geq\gamma}B(\delta)\};\\
N_{\gamma} &= \{s\in N \text{ such that } \min(\text{support } s)>\gamma\}\\
&= \{s\in M \text{ such that } s\in\mathbf{H}_{\delta>\gamma}B(\delta)\}.
\end{aligned}$$

So $B(M,\gamma)=\{s+M_\gamma\ ;\ s\in M^\gamma\}$ and $B(N,\gamma)=\{s+N_\gamma\ ;\ s\in N^\gamma\}$ which are canonically isomorphic to $B(\gamma)$ as modules.

3. **Definition 0.4** *Let* $(\Gamma,\leq)$ *be a totally ordered set. We say that* Γ *is* ***well-ordered*** *if any non empty subset* $A\subseteq\Gamma$ *has a least element.*
Given a well-ordered set $(\Gamma,\leq)$*, its* ***order type*** $ot(\Gamma)$ *is defined to be a fixed representative of its equivalence class by ordered set isomorphism, and is called an* ***ordinal number****. In particular, the order type of the set of natural numbers is denoted by* $ot(\mathbb{N}):=\omega$*. It is the smallest infinite ordinal number.*

(a) Given 2 ordered sets $(A,\leq)$ and $(B,\leq)$, one defines the **sum of ordered sets**: $(A,\leq_A)+(B,\leq_B)=A+B:=(A\sqcup B,\leq_+)$ ($\sqcup$ = disjoint union) such that for any

$$c_1,c_2\in A\sqcup B,\ c_1\leq_+ c_2 \Leftrightarrow \begin{cases}\text{either } (c_1,c_2\in A \text{ and } c_1\leq_A c_2)\\ \text{or } (c_1\in A,\ c_2\in B \text{ and } c_1<_+ c_2)\\ \text{or } (c_1,c_2\in B \text{ and } c_1\leq_B c_2).\end{cases}.$$

Consider a nonempty subset $C\subseteq A+B$. As a set, $C=(C\cap A)\sqcup(C\cap B)$ with at least one of the two $C_A=C\cap A$ and $C_B=C\cap B$ which is nonempty. Whenever it is nonempty, as a subset of a well-ordered set C_A, respectively C_B, has a least element, say c_A, respectively c_B. Then, whenever B, respectively A, is empty, c_A, respectively c_B, is the least element of C itself. If A and B are nonempty, we have $c_A<B$ by definition of the ordering on $A+B$. So c_A is the least element of C.

(b) Suppose that A and B are well-ordered sets. Denote $\alpha:=ot(A)$ and $\beta:=ot(B)$.

One defines the **sum of ordinals** as

$$\alpha + \beta := ot(A + B).$$

Given any other well-ordered sets A' and B' with $ot(A') = \alpha$ and $ot(B') = \beta$, we just have to show that $ot(A' + B') = \alpha + \beta$, i.e. $A' + B'$ is order isomorphic to $A + B$. Consider some isomorphisms of ordered sets $\phi : A \to A'$ and $\psi : B \to B'$. We define the map $\Phi : A + B \to A' + B'$ such that for any $c \in A + B$, if $c \in A$, $\Phi(c) := \phi(c) \in A'$, and if $c \in B$, $\Phi(c) := \psi(c) \in B'$. Then it is easy to show that Φ is an isomorphism of ordered sets.
Concerning the non commutativity, we consider $1 + \omega$ and $\omega + 1$. We have $ot(1 + \omega) = \omega$ which has no greatest element, whereas the 1 on the right side is the greatest element of $\omega + 1$: the two sets $1 + \omega$ and $\omega + 1$ cannot be order isomorphic.

(c) Define for any $k,l \in \mathbb{N}^*$, $a_{k,l} := k - \frac{1}{l}$. Then the set $Q_n := \bigcup_{k=1}^{n} \bigcup_{l \in \mathbb{N}^*} \{a_{k,l}\}$ endowed with the retriction of the ordering on $\mathbb{Q}$, is a totally ordered set with order type $\omega.n$.

(d) Given 2 ordered sets $(A, \leq_A)$ and $(B, \leq_B)$, one defines the **product of ordered sets**:

$(B, \leq_B).(A, \leq_A) = B.A := (A \times B, \leq_{\text{lex}})$ such that $\leq_{\text{lex}}$ is the lexicographic ordering, i.e. for any $(a_1,b_1),(a_2,b_2) \in A \times B$,

$$(a_1,b_1) \leq_{\text{lex}} (a_2,b_2) \Leftrightarrow \begin{cases} a_1 <_A a_2 \\ \text{or } a_1 = a_2 \text{ and } b_1 \leq_B b_2 \end{cases}.$$

Consider a nonempty subset $C \subset B.A$. Then it can be written as $C = \bigcup_{a \in C_A} \{a\} \times C_{a,B}$, with $\emptyset \neq C_A \subset A$ and for any $a \in C_A$, $\emptyset \neq C_{a,B} \subset B$. As nonempty subsets of well-ordered sets, C_A has a least element a_C, and $C_{a_C,B}$ has a least element b_C. Then (a_C,b_C) is the least element of $B.A$.

(e) Suppose that A and B are well-ordered sets. Denote $\alpha := ot(A)$ and $\beta := ot(B)$, one defines the **product of ordinals**:

$$\alpha.\beta := ot(A.B).$$

Given any other well-ordered sets A' and B' with $ot(A') = \alpha$ and $ot(B') = \beta$, we just have to show that $ot(B'.A') = \beta.\alpha$, i.e. $B'.A'$ is order isomorphic to $B.A$. Consider some isomorphisms of ordered sets $\phi : A \to A'$ and $\psi : B \to B'$. We define the map $\Psi : B.A \to B'.A'$ such that for any $(a,b) \in B.A$, $\Psi(a,b) := (\phi(a),\psi(b)) \in B'.A'$. Then Ψ is clearly bijective. Moreover, take any $(a_1,b_1) > (a_2,b_2) \in B.A$, then either $a_1 > a_2$ which would imply that $\phi(a_1) > \phi(a_2)$ and so $\Psi(a_1,b_1) > \Psi(a_2,b_2) \in B'.A'$, or $a_1 = a_2$ and $b_1 > b_2$ which would imply that $\phi(a_1) = \phi(a_2)$ and $\psi(b_1) > \psi(b_2)$, and also $\Psi(a_1,b_1) > \Psi(a_2,b_2) \in B'.A'$.
For the non commutativity, consider $2.\omega$ and $\omega.2$. Consider the set $\mathbb{N} \times \{0,1\}$ endowed with the lexicographic ordering. It is clearly order isomorphic to $2.\omega$. Then the map $f : \mathbb{N} \times \{0,1\} \to \mathbb{N}$ such that $f(n,\epsilon) := 2n + \epsilon$ (where $\epsilon \in \{0,1\}$),

is an isomorphism of orderings. Thus $ot(2.\omega) = \omega$. But $\omega.2 = \omega + \omega$ cannot be order isomorphic to ω (it contains $\omega + 1$ as an initial segment).

(f) Consider the map

$$\begin{array}{rccl} f: & \mathbb{N}^* \times \mathbb{Q}^* & \to & \mathbb{Q} \\ & (k,l) & \to & k - \dfrac{1}{l}. \end{array}$$

and we define by induction on $n \in \mathbb{N}^*$,

for any $n \in \mathbb{N}^*$, for any tuple $(k_1, \ldots, k_n) \in [\mathbb{N} * \times (\mathbb{N}^* \setminus \{1\})^{n-2} \times \mathbb{N}^*]$

$$\begin{cases} a_{(k_1)} & := & k_1 \\ a_{(k_1,\ldots,k_n)} & := & f(k_n, a_{(k_1,\ldots,k_{n-1})}) \\ & = & k_n - \dfrac{1}{a_{(k_1,\ldots,k_{n-1})}}. \end{cases}$$

Then for any $n \in \mathbb{N}^*$, we define $Q_n := \bigcup_{(k_1,\ldots,k_n) \in [\mathbb{N}^* \times (\mathbb{N}^* \setminus \{1\})^{n-2} \times \mathbb{N}^*]} \{a_{(k_1,\ldots,k_n)}\}$. Then Q_n endowed with the ordering of $\mathbb{Q}$ is order isomorphic to ω^n.