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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 12 - Lösung

Definition 0.1 *Let R be a real closed field. Let $A \subseteq R^n$ be a semi-algbraic set.*
*(i) A **semialgebraic path** in A is a continuous semialgebraic map $\alpha : (0,1) \to A$.*
*(ii) The set A is **semialgebraically compact** if for every path $\alpha : (0,1) \to A$, $\lim_{t\to 0^+} \alpha(t)$ exists and is in A.*

1. **Theorem 0.2 (semialgebraic choice = Semi-algebraische Auswahl)** *Let A and B be semialgebraic sets and $f : A \to B$ be a surjective semialgebraic map. Then f has a semialgebraic inverse, i.e. there is a semialgebraic map $g : B \to A$ with $f(g(y)) = y$ for any $y \in B$.*
Proof. We can suppose $A \subset R^m$ and $B \subset R^n$ semialgebraic subsets. Decompose f as
$$A \to^{\gamma} \Gamma(f) \subset R^{m+n} \to^{\pi} B$$
where $\gamma(\underline{x}) = (\underline{x}, f(\underline{x}))$ for any $\underline{x} \in A$ and $\pi(\underline{x},\underline{y}) = \underline{y}$ for any $(\underline{x},\underline{y}) \in R^{m+n}$.
Since γ is bijective, it suffices to find a semialgebraic section for π. In other words, we consider a semialgebraic set $\tilde{A} \subseteq R^{m+n}$ and the semialgebraic map π. Then proceed by induction on n: the case $n = 1$ is given by the exercise 4 of Blatt 11.

2. **Corollary 0.3 (Curve Selection Lemma: unbounded case)** *Let $A \subseteq R^n$ be an unbounded semialgebraic set. Then there exists a semialgebraic path $\alpha :]0,1[\to A$ with $\lim_{t\to 0} \|\alpha(t)\| = +\infty$.*
Proof. Consider the stereographic projection $p : S_n(\underline{0},1) \setminus \{\infty\} \to R^n$, which is a homeomorphism, and its inverse $p^{-1} : R^n \to S_n(\underline{0},1) \setminus \{\infty\}$. From Exercise 4 Part (b) of Blatt 10, since A is unbounded, we know that $\infty \in \overline{p^{-1}(A)}$. Now, applying the Curve Selection Lemma to $p^{-1}(A)$, there exists a semi-algebraic continuous map $\beta : [0,1[\to S^n$ with $\beta(]0,1[) \subset p^{-1}(A)$ and $\beta(0) = \infty$. Then consider the path $\alpha := p \circ \beta :]0,1[\to A$.

3. (a)

Lemma 0.4 *Let A and B be semialgebraic sets and $f : A \to B$ be a semialgebraic map. Let $\beta :]0,1[\to B$ be a semialgebraic path in B with $\beta(]0,1[) \subseteq f(A)$. Then there exists $c \in R$ with $0 < c < 1$ and there exists a semialgebraic path $\alpha :]0,c[\to A$ such that $\beta(t) = f(\alpha(t))$ for any $t \in]0,c[$.*

Proof. From the Theorem of Semialgebraic Choice here above, there exists a semialgebraic $\alpha :]0,1[\to A$ such that $f \circ \alpha = \beta$. Now, from Exercise 2.(b) of Blatt 11, the map α is continuous for all but finitely many points of $]0,1[$. Then consider $c \in]0,1[$ the smallest point for which α is not continuous. So it is continuous on $]0,c[$.

(b) Let A be a semialgebraically compact set and $f : A \to R$ a semialgebraic function. Using the cited result, $f(A)$ is sa compact in R. So, by the Theorem on the characterisation of sa compact sets, $f(A)$ is closed and bounded in R. But any semialgebraic set of R is a finite union of points and intervals. So $f(A)$ is of the form $\bigcup_{i=0}^{k} [a_i,b_i]$ for some $k \in \mathbb{N}$ with $a_i,b_i \in R$ for all $i = 1,\ldots,k$. Thus it has a least element and a greatest element.

4. (a) Let $A \subseteq R^n$ be a semialgebraic set, $x \in A$. For any non negative integer k, the open ball $B_n(x,1/2^k)$ is a semialgebraic neighborhood of x in R^n. So for any k, $U_k := B_n(x,1/2^k) \cap A$ is semi-algebraic and non empty since it contains x. Thus it has dimension d_k.

Underline that for any semialgebraic sets A and B, if $A \subset B$ then $\dim A \leq \dim B$ (follows directly from the definition of the semialgebraic dimension). Thus, since $U_{k+1} \subset U_k$ for any k, we have $d_{k+1} \leq d_k$. But such a decreasing sequence of non negative integers needs to stabilize: $\exists k_0, \forall k \geq k_0, d_k = d_{k_0}$. Then put $U := U_{k_0}$ and $d := d_{k_0}$.

The integer d is called the **dimension of** A **at** x and denoted by $\dim_x A$.

(b) Consider a cell decomposition $A = \dot{\bigcup}_{i=1}^{m} C_i$ (disjoint union) of A, i.e. for each i, C_i is isomorphic to $(0,1)^{d_i}$ for some non negative integer d_i. Then $d := \dim A = \max_{i=1,\ldots,m}(d_i)$ by definition. Say $d = d_1$ for instance.

For any $x \in C_1$, there exist an open neighborhood U of x in R^n and a nonnegative integer d'_1 such that, for every semialgebraic neighborhood $V \subset U$ of x in R^n, $\dim(V \cap A) = d'_1$. We want to show that $d'_1 = d_1$. First, we note that $\dim(V \cap A) = d'_1 \leq d_1 = \dim A$, since $V \cap A \subset A$.

Consider $U_1 := U \cap C_1$ which is an open neighborhood of x in $C_1 \subset A$. Since C_1 is homeomorphic to $(0,1)^{d_1}$, U_1 must contain some open ball $B_{d_1}(x,r)$. Then, up to a restriction of U to $B_n(x,r)$, we obtain that for any semi-algebraic neighborhood $V \subset U$, $\dim V \cap A = d_1$, which means that $d'_1 = d_1$.

(c) Denote $D := \{x \in A; \dim_x A = \dim A\}$ and consider $x \in \overline{D}$ the closure of D.

For any open neighborhood $V \in R^n$ of x, it contains a point $y \in D$. But, there exists an open neighborhood U_y of y such that, for any semi-algebraic neighborhood $V_y \subset U_y$ of y, $\dim(V_y \cap A) = d$. So, fix an open neighborhood U of x. For any open semi-algebraic neighborhood $V \subset U$ of x, we have $\dim(V \cap A) = \dim(V_y \cap A) = d$. Thus $x \in D$.