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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 1

This assignment is somewhat short, due to the fact that there are only five days until Tuesday 27 October. For the next sessions, there will be 7 days, from Tuesday to Tuesday. Exercises normally will be collected in one of the mailboxes of the Mathematics department, or during the break of Tuesday's lecture.

1. (i) Show that the relation

$$(a,b) \le (c,d) \Leftrightarrow a \le c \text{ and } b \le d \text{ in } \mathbb{R}$$

defines a partial order on $\mathbb{R} \times \mathbb{R}$, which is not total.
This partial order is called the *pointwise order*.

(ii) Show that the relation

$$(a,b) \le (c,d) \Leftrightarrow a < c \text{ or } (a = c \text{ and } b \le d) \text{ in } \mathbb{R}$$

defines a total order on $\mathbb{R} \times \mathbb{R}$.
This total order is called the *lexicographic order*.

2. Let $(F, \le)$ be an ordered field. For any a,b,c in F, show that:

(i) $a \le b \Leftrightarrow 0 \le b - a$
(ii) $0 \le a^2$
(iii) $a \le b$ *and* $0 \le c \Rightarrow ac \le bc$
(iv) $0 < a \le b \Rightarrow 0 < \frac{1}{b} \le \frac{1}{a}$
(v) $0 < ab \Leftrightarrow 0 < \frac{a}{b}$
(vi) $0 < n, \forall n \in \mathbb{N}$

3. Let $(F, \le)$ be an ordered field. For any a,b in F, show that:

$$|a| > |b| \Rightarrow sign(a+b) = sign(a)$$

4. A totally ordered set $(\Gamma, \leq)$ is *Dedekind complete* if for any pair of non empty subsets L and U of Γ such that $L \leq U$, there exists $\gamma \in \Gamma$ such that $L \leq \gamma \leq U$.
A pair of subsets (L,U) of Γ is called a *free Dedekind cut* or a *gap* if:
(i) (L,U) is a *Dedekind cut* : $L \cap U = \emptyset$, $L \cup U = \Gamma$ and $L < U$;
(ii) L and U are non empty, L has no last element and U no least element.
Show that Γ is Dedekind complete if and only if Γ has no free Dedekind cuts.