REAL ALGEBRAIC GEOMETRY LECTURE NOTES (30: 11/02/10)

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Contents

1. Convex valuations

Let K be a non-archimedean ordered field. Let v be its non-trivial natural valuation with valuation ring R_v and valuation ideal I_v.

Remark 1.1.
(1) R_v/I_v is archimedean.

(2) R_v is the convex hull of $\mathbb{Q}$ in K.

Let w be any valuation of K with valuation ring R_w, valuation ideal I_w and residue field $K_w := R_w/I_w$.

Definition 1.2. We say that w **is compatible with the order** if $\forall\, a, b \in K$

$$0 < a \leqslant b \ \Rightarrow\ w(a) \geqslant w(b).$$

Compatible valuations are also called **convex valuations**.

Example 1.3. The natural valuation is compatible with the order.

Remark 1.4. We recall that a subset C of a totally ordered set X is said to be **convex** if $\forall\, c_1, c_2 \in C$ and $x \in X$:

$$c_1 < x < c_2 \ \Rightarrow\ x \in C.$$

If C is a subgroup of an ordered abelian group A, equivalently C is convex if and only if $\forall\, c \in C$ and $a \in A$:

$$0 < a < c \ \Rightarrow\ a \in C.$$

Proposition 1.5. *(Characterization of convex valuations). The following are equivalent:*

(1) *w is compatible with the order of K.*

(2) *R_w is convex.*

(3) *I_w is convex.*

(4) *$I_w < 1$.*

(5) *$1 + I_w \subseteq K^{>0}$.*

(6) *The residue map*

$$\begin{array}{ccc} R_w & \longrightarrow & R_w/I_w \\ a & \mapsto & a + I_w \end{array}$$

induces an ordering on Kw given by

$$a + I_w \geqslant 0 \;\Leftrightarrow\; a \geqslant 0.$$

(7) *The set*

$$\mathcal{U}_w^{>0} := \{a \in K : w(a) = 0 \;\wedge\; a > 0\}$$

of positive units is a convex subgroup of $(K^{>0}, \cdot, 1, <)$.

Proof. (1) $\Rightarrow$ (2). $0 \leqslant a \leqslant b \in R_w \Rightarrow w(a) \geqslant w(b) \geqslant 0$.

(2) $\Rightarrow$ (3). Let $a, b \in K$ with $0 < a < b \in I_w$. Since $w(b) > 0$, it follows that $w(b^{-1}) = -w(b) < 0$ and then $b^{-1} \notin R_w$.

Therefore also $a^{-1} \notin R_w$, because $0 < b^{-1} < a^{-1}$ and R_w is convex by assumption. Hence $w(a) > 0$ and $a \in I_w$.

(3) $\Rightarrow$ (4). Otherwise $1 \in I_w$ but $w(1) = 0$, contradiction.

(4) $\Rightarrow$ (5). Clear.

□

2. Comparison of convex valuations

Let w and w' be valuations on K. We say that w' is **finer** than w or w is **coarser** than w' if w' has a smallest valuation ring, i.e. if

$$R_{w'} \subsetneq R_w.$$

Lemma 2.1.

(1) *$R_{w'} \subsetneq R_w$ if and only if $I_w \subsetneq I_{w'}$.*

(2) *If w' is convex and $R_{w'} \subsetneq R_w$, then w is also convex.*

(3) *The set $\mathcal{R}$ of all convex valuation rings R_w is totally ordered by inclusion.*

(4) *The natural valuation is the finest convex valuation, i.e.*

$$R_v \subsetneq R_w,$$

for every convex valuation $w \neq v$.

3. The rank of ordered fields

Definition 3.1. Let K be an ordered field with natural valuation v. The set $\mathcal{R}$ of all valuation rings R_w of convex valuations $w \neq v$ is called the **rank** of K.

Examples 3.2.

- The rank of an archimedean ordered field is empty since its natural valuation is trivial.
- The rank of the rational function field $K = \mathbb{R}(t)$ with any order is a singleton.

4. Convex valuations and convex subgroups

Notation 4.1. For simplicity we denote by $w(K)$ the value group of a valuation w on K (even if $w(0) = \infty$).

To every convex valuation w on K we associate a convex subgroup G_w of $v(K)$, namely

$$G_w := \{v(a) : a \in K \,\wedge\, w(a) = 0\} = v(\mathcal{U}_w^{>0}).$$

Proposition 4.2.

$$w(K) \;\cong\; v(K)/G_w$$

canonically.

Proof. The map

$$\begin{aligned} v(K)/G_w &\longrightarrow w(K) \\ v(a) + G_w &\mapsto w(a) \end{aligned}$$

is well defined and an isomorphism. $\square$

We call G_w **the convex subgroup associated to** w. Note that the convex subgroup G_v associated to the natural valuation v is

$$G_v = \{0\}.$$

Conversely, given a convex subgroup G_w of $v(K)$ we define a map:

$$\begin{array}{rcl} w\colon K & \longrightarrow & v(K)/G_w \cup \{\infty\} \\ 0 \neq a & \mapsto & v(a) + G_w \\ 0 & \mapsto & \infty \end{array}$$

Then w is a convex valuation with $v(\mathcal{U}_w^{>0}) = G_w$. We call w the convex valuation associated to G_w.

We have proved the following theorem:

Theorem 4.3. *There is a bijection between the set of convex valuations on an ordered field K and the set of convex subgroups of the value group $v(K)$ associated to the natural valuation v.*