

REAL ALGEBRAIC GEOMETRY LECTURE NOTES
(26: 28/01/10)

SALMA KUHLMANN

CONTENTS

1. Pseudo-completeness	1
------------------------	---

1. PSEUDO-COMPLETENESS

Let (V, v) be a valued Q -vector space. We recall that

- (V, v) is said to be **maximally valued** if it admits no proper immediate extension.
- (V, v) is **pseudo-complete** if every pseudo-convergent sequence in V has a pseudo-limit in V .

Theorem 1.1. *(V, v) is maximally valued if and only if (V, v) is pseudo-complete.*

We prove only one implication:

(V, v) pseudo-complete \Rightarrow (V, v) maximally valued.

This implication follows from the following:

Proposition 1.2. *Let (V, v) be an immediate extension of (V_0, v) . Then any element in V which is not in V_0 is a pseudo-limit of a pseudo-Cauchy sequence of elements of V_0 , without a pseudo-limit in V_0 .*

Proof. Let $z \in V \setminus V_0$. Consider the set

$$X = \{v(z - a) : a \in V_0\}.$$

Since $z \notin V_0$, $\infty \notin X$.

We show that X cannot have a maximal element. Otherwise, assume $a_0 \in V_0$ and $v(z - a_0)$ maximal in X . Since the extension is immediate, by Lemma 2.2 of Lecture 24 there is $a_1 \in V_0$ such that $v(z - a_0 - a_1) > v(z - a_0)$. So $a_0 + a_1 \in V_0$ and $v(z - (a_0 + a_1)) > v(z - a_0)$, contradiction. Then X has no greatest element.

Select from X a well ordered cofinal subset $\{\alpha_\rho\}_{\rho \in \lambda}$. Since the set X has no greatest member, also $\{\alpha_\rho\}_{\rho \in \lambda}$ does not have a last term (see Lemma 4.3 of Lecture 25).

For every $\rho \in \lambda$ choose an element $a_\rho \in V_0$ with

$$v(z - a_\rho) = \alpha_\rho.$$

The identity

$$a_\sigma - a_\rho = (z - a_\rho) - (z - a_\sigma)$$

together with the inequality

$$v(z - a_\rho) < v(z - a_\sigma) \quad (\forall \rho < \sigma \in \lambda)$$

imply

$$(*) \quad v(a_\sigma - a_\rho) = v(z - a_\rho).$$

Then $\{a_\rho\}_{\rho \in \lambda}$ is pseudo-convergent with z as a pseudo-limit.

Suppose now that $\{a_\rho\}_{\rho \in \lambda}$ had a further limit $z_1 \in V_0$.

Then by Lemma 3.6 of Lecture 25 we have

$$v(z - z_1) > v(a_\sigma - a_\rho).$$

Combining this with $(*)$ we get

$$v(z - z_1) > v(z - a_\rho) = \alpha_\rho \quad \forall \rho \in \lambda$$

and this is a contradiction, since $\{\alpha_\rho\}_{\rho \in \lambda}$ is cofinal in X . \square

Theorem 1.3. *Suppose that*

- (i) V_i and V'_i are Q -valued vector spaces and V'_i is an immediate extension of V_i , for $i = 1, 2$.
- (ii) h is an isomorphism of valued vector spaces of V_1 onto V_2 .
- (iii) V'_2 is pseudo-complete.

Then there exists an embedding h' of valued vector spaces of V'_1 in V'_2 such that h' extends h .

Moreover h' is an isomorphism of valued vector spaces of V'_1 onto V'_2 if and only if V'_1 is pseudo-complete.

Proof. The picture is the following:

$$\begin{array}{ccc} V'_1 & \xrightarrow{h'} & V'_2 \\ \text{immediate} \Big| & & \Big| \text{immediate} \\ V_1 & \xrightarrow[h]{\sim} & V_2 \end{array}$$

By Zorn's Lemma, let

$$\begin{aligned} V_1 &\subseteq M_1 \subseteq V'_1, \\ V_2 &\subseteq M_2 \subseteq V'_2 \end{aligned}$$

and g a valuation isomorphism of M_1 onto M_2 extending h . We shall show how to extend g to V'_1 .

Let $y_1 \in V'_1 \setminus M_1$. Since V'_1 is an immediate extension of M_1 there exists a pseudo-convergent sequence

$$S = \{a_\rho\}_{\rho \in \lambda}$$

of M_1 without a pseudo-limit in M_1 but with a pseudo-limit $y_1 \in V'_1$.

Consider

$$g(S) = \{g(a_\rho)\}_{\rho \in \lambda}$$

Since g is a valuation preserving isomorphism, $g(S)$ is a pseudo-convergent sequence of M_2 without a pseudo-limit in M_2 but with pseudo-limit $y_2 \in V'_2$, because V'_2 is pseudo-complete.

Let $M'_i = \langle M_i, y_i \rangle$, for $i = 1, 2$, and denote by g' the unique Q -vector space isomorphism of the linear space M'_1 onto the linear space M'_2 extending g and such that $g'(y_1) = y_2$.

We show that g' is valuation preserving: let

$$y = x + qy_1 \quad x \in M_1 \quad q \in Q \setminus \{0\}$$

be an arbitrary element of $M'_1 \setminus V_1$. The set

$$S(y) = \{x + qa_\rho\}_{\rho \in \lambda}$$

is a pseudo-convergent sequence in M_1 with pseudo-limit $y \in M'_1$ and 0 is not a pseudo-limit (otherwise $-x/q \in M_1$ would be a pseudo-limit of S).

It follows that (since $y = x + qy_1$ is a pseudo-limit for the sequence $x + qa_\rho$ which does not have 0 as a pseudo-limit)

$$v(y) = \text{Ult } S(y)$$

similarly

$$v(g'(y)) = \text{Ult } S(g'(y))$$

where

$$S(g'(y)) = \{g'(x) + qg'(a_\rho)\}_{\rho \in \lambda}$$

is a pseudo-convergent sequence of M_2 with limit $g'(y) \in M'_2$.

Now $g'|_{M_1} = g$ is valuation preserving from M_1 to M_2 . So we have

$$\text{Ult}(S(y)) = \text{Ult}(S(g'(y)))$$

hence

$$v(y) = v(g'(y))$$

as required. □

Proposition 1.4. $\mathbb{H}_{\gamma \in \Gamma} B(\gamma)$ is pseudo-complete.

Proof. Let $\{a_\rho\}_{\rho \in \lambda}$ be pseudo-Cauchy. Recall that

$$\gamma_\rho = v(a_\rho - a_{\rho+1})$$

is strictly increasing. Define $x \in \mathbb{H}_{\gamma \in \Gamma} B(\gamma)$ by

$$x(\gamma) = \begin{cases} a_\rho(\gamma) & \text{if } \gamma < \gamma_\rho \\ 0 & \text{otherwise.} \end{cases}$$

It is well defined because if $\rho_1 < \rho_2$, $\gamma < \gamma_{\rho_1}$ and $\gamma < \gamma_{\rho_2}$, then

$$v(a_{\rho_1} - a_{\rho_2}) = \gamma_{\rho_1}$$

and then

$$a_{\rho_1}(\gamma) = a_{\rho_2}(\gamma).$$

We show now that $\text{support}(x)$ is well ordered.

Let $A \subseteq \text{support}(x)$, $A \neq \emptyset$ and $\gamma_0 \in A$. Then $\exists \rho$ such that $\gamma_0 < \gamma_\rho$ and $x(\gamma_0) = a_\rho(\gamma_0)$ with $\gamma_0 \in \text{support}(a_\rho)$.

Consider

$$A_0 := \{\gamma \in A : \gamma \leq \gamma_0\}.$$

Note that since $x(\gamma) = a_\rho(\gamma)$ for $\gamma \leq \gamma_0$ it follows that $A_0 \subseteq \text{support}(a_\rho)$ which is well ordered, so $\min A_0$ exists in A_0 and it is the least element of A .

We now conclude by showing that x is a pseudo-limit. From definition of x we have

$$v(x - a_\rho) \geq \gamma_\rho = v(a_{\rho+1} - a_\rho) \quad \forall \rho.$$

If $v(x - a_\rho) > v(a_\rho - a_{\rho+1})$, then

$$v(x - a_{\rho+1}) = v(x - a_\rho + a_\rho - a_{\rho+1}) = v(a_\rho - a_{\rho+1}) = \gamma_\rho$$

but

$$v(x - a_{\rho+1}) \geq \gamma_{\rho+1} > \gamma_\rho,$$

contradiction. □

As a corollary to the general embedding theorem and this proposition we get Hahn's embedding's theorem.