REAL ALGEBRAIC GEOMETRY LECTURE NOTES (21: 12/01/10)

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Contents

Let R be a real closed field.

1. Thom's Lemma

Lemma 1.1. *Let $A \subset R$ be a semialgebraic set and $\varphi\colon A \to R$ a semialgebraic function. Then exists $f \in R[\mathrm{x}, \mathrm{y}]$, $f \neq 0$, such that*

$$\forall\, x \in A \qquad f(x, \varphi(x)) = 0 \qquad (f \text{ vanishes on the graph of } \varphi).$$

Proof. The graph of φ $\Gamma(\varphi) = \{(x, \varphi(x)) : x \in A\} \subset R^2$ is a semialgebraic set, so it is a finite union of sets of the form

$$\{(x, y) \in R^2 : f_i(x, y) = 0,\ i = 1, \dots, l\ g_j(x, y) > 0,\ j = 1, \dots, m\}$$

with at least one among the $f_i \neq 0$, otherwise $\Gamma(\varphi)$ would contain an open subset of R^2, contradiction.

Now take f to be the product of these nonzero polynomials. □

Proposition 1.2. *(Thom's Lemma) Let $\{f_1, \dots, f_s\}$ be a family of non-zero polynomials in $R[X]$ closed under derivation. Let $\varepsilon\colon \{1, \dots, s\} \to \{-1, 0, 1\}$ be a sign function. Set*

$$A_\varepsilon := \{x \in R : \operatorname{sign}(f_k(x)) = \varepsilon(k),\ k = 1, \dots, s\}.$$

Denote by $A_{\bar\varepsilon}$ the semialgebraic subset of R obtained by relaxing the strict inequalities in A_ε, i.e. :

$$A_{\bar\varepsilon} := \bigcap_{k=1}^{s} \{x \in R : \operatorname{sign}(f_k(x)) \in \bar\varepsilon(k)\}.$$

where $\bar\varepsilon$ is defined as follows:

$$\bar{0} = \{0\} \qquad -\bar{1} = \{-1, 0\} \qquad \bar{1} = \{0, 1\}.$$

Then

(*i*) *either A_ε is empty, or A_ε is a point, or A_ε is a non-empty open interval (if A_ε is empty or a point, then $\varepsilon(k)=0$ for some k; if A_ε is a non-empty open interval then $\varepsilon(k)=\pm 1$ for every k);*

(*ii*) *if A_ε is non-empty then its closure is $A_{\bar\varepsilon}$ (which is either a point or a closed interval different from a point and the interior of this interval is A_ε);*

(*iii*) *if A_ε is empty then $A_{\bar\varepsilon}$ is either empty or a point.*

Proof. By induction on s. The Lemma holds trivially for $s=0$. Let $f_1,\dots,f_s,f_{s+1}\in R[\mathrm{x}]\setminus\{0\}$ be polynomials such that if $f'_k\neq 0$, then $f'_k\in\{f_1,\dots,f_{s+1}\}$. Without loss of generality we assume that $\deg(f_{s+1})=\max\{\deg(f_k):1\leqslant k\leqslant s+1\}$.

Let $\varepsilon'\colon\{1,\dots,s,s+1\}\to\{-1,0,1\}$ and $\varepsilon\colon\{1,\dots,s,\}\to\{-1,0,1\}$ the restriction.

Note that

$$A_{\varepsilon'}=A_\varepsilon\cap\{x\in R:\operatorname{sign}(f_{s+1}(x))=\varepsilon'(s+1)\}.$$

By induction A_ε is empty, a point, or an interval.

If A_ε is empty or a point, then obviously so is $A_{\varepsilon'}$ and the other property follows immediately by induction hypothesis on A_ε.

Assume A_ε is an interval. Now $f'_{s+1}=0$ or $f'_{s+1}\in\{f_1,\dots,f_s\}$. So by definition of A_ε, f'_{s+1} has constant sign on A_ε. Therefore f_{s+1} is either strictly increasing, or strictly decreasing or constant on A_ε.

Consider $A_\varepsilon=(a,b)$ There are three cases depending on $\varepsilon'(s+1)$:

Case 1. $A_{\varepsilon'}=\{x\in(a,b):f_{s+1}(x)>0\}$.
Case 2. $A_{\varepsilon'}=\{x\in(a,b):f_{s+1}(x)<0\}$.
Case 3. $A_{\varepsilon'}=\{x\in(a,b):f_{s+1}(x)=0\}$.

If $A_{\varepsilon'}=\emptyset$ there is nothing to prove.

Assume $A_{\varepsilon'}\neq\emptyset$. If f_{s+1} is constant on A_ε then f_{s+1} is a constant polynomial $f_{s+1}(x)=c\neq 0$. So $A_{\varepsilon'}$ is empty or $A_{\varepsilon'}=(a,b)$ depending on whether $\operatorname{sign}(c)=\varepsilon'(s+1)$.

Assume now f_{s+1} strictly increasing on A_ε and $A_{\varepsilon'}=\{x\in(a,b):f_{s+1}(x)>0\}\neq\emptyset$. Let $x_0=\inf\{x\in(a,b):f_{s+1}(x)>0\}$. Since f_{s+1} is strictly increasing it follows that $f_{s+1}(x)>0\ \forall x\in(a,b)$ with $x>x_0$. So $A_{\varepsilon'}=(x_0,b)$ and its closure is $[x_0,b]=A_{\bar\varepsilon'}$. The other cases are treated similarly. □

2. Semialgebraic path connectedness

Definition 2.1. Let $A\subseteq R^n$ be a semialgebraic set.

(1) A **semialgebraic path** in A is a continuous semialgebraic map
$$\alpha\colon I \longrightarrow A,$$
where I is either $[0,1]$ or $]0,1[$.

(2) Let $x, y \in A$. We say that x is semialgebraic path connected to y if there exists a semialgebraic path in A
$$\alpha\colon [0,1] \longrightarrow A$$
with $\alpha(0) = x$ and $\alpha(1) = y$.

Remark 2.2. Note that "x is semialgebraic path connected to y" is an equivalence relation on A:

To see simmetry observe that if α is a path from x to y then
$$\alpha^*(t) := \alpha(1-t)$$
defines a path from y to x.

To see transitivity observe that if α is a path from x to y and β is a path from y to z, then
$$\gamma(t) := \begin{cases} \alpha(2t) & 0 \leqslant t \leqslant 1/2 \\ \beta(2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases}$$
is a path from x to z.

(3) A is **semialgebraic path connected** if any two points in A are semialgebraic path connected.

Proposition 2.3. *Let A be a semialgebraic set. Then*

A is semialgebraic connected $\iff$ A is semialgebraic path connected.

Proof.

($\Rightarrow$) Suppose A is a semialgebraic connected set and let
$$A = \bigcup_{i=1}^{n} C_i$$
a semialgebraic cell decomposition of A (so each C_i is semialgebraic path connected). Then we have seen that there is an equivalence relation on $\{C_i : i = 1, \ldots, n\}$ given by:

$$C_i \sim C_j \Leftrightarrow \exists\, C_{i_0}, \ldots, C_{i_q} \text{ such that } C_{i_0} = C_i,\ C_{i_q} = C_j \text{ and } C_{i_k} \cap \bar{C}_{i_{k+1}} \neq \emptyset \text{ or } \bar{C}_{i_k} \cap C_{i_{k+1}} \neq \emptyset\ \ \forall\, 0 \leqslant k < q,$$

such that the equivalence classes with respect to this equivalence relation are the semialgebraic connected component of S. Since A is semialgebraic connected there is only one equivalence class.

Claim 1. If C is a semialgebraic path connected set, also the closure $\bar{C}$ of C is semialgebraic path connected (it is an immediate consequence of the Curve Selection Lemma).

Claim 2. If $A_1, A_2 \subseteq R^n$ are semialgebraic path connected with $A_1 \cap A_2 \neq \emptyset$, then $A_1 \cup A_2$ is semialgebraic path connected.

So let $x, y \in A$. We want to find a semialgebraic path in A joining x and y. Let $x \in C_i$ and $y \in C_j$ and $C_{i_0}, \ldots, C_{i_q}$ as above. For every $0 \leqslant k < q$, let $a_k \in C_{i_k} \cap \bar{C}_{i_{k+1}}$ or $a_k \in \bar{C}_{i_k} \cap C_{i_{k+1}}$. By Claim 1 and Claim 2 we can find semialgebraic paths joining a_k with a_{k+1} for every $0 \leqslant k < q$ and conclude joining x with a_0 (since $C_i = C_{i_0}$ is semialgebraic path connected) and a_{q-1} with y (since $C_j = C_{i_q}$ is semialgebraic path connected).

($\Leftarrow$) **Claim.** If A is path connected then A is connected.

Suppose for a contradiction that A is a disjoint union of non-empty open sets A_1 and A_2. Take $x \in A_1$, $y \in A_2$ and $\varphi : [0,1] \to A$ a continuous function such that $\varphi(x) = 0$ and $\varphi(y) = y$ (it exists because A is path connected).

Now consider $X_1 := [0,1] \cap \varphi^{-1}(A_1)$ and $X_2 := [0,1] \cap \varphi^{-1}(A_2)$. Then X_1 and X_2 disconnect $[0,1]$, contradiction.

So we have:

A semialg. path conn. $\Rightarrow$ A path conn. $\Rightarrow$ A conn. $\Rightarrow$ A semialg. conn.

□

The semialgebraic assumption is essential to prove ($\Rightarrow$), as the following example shows:

Example 2.4. Let $\Gamma = \{(x,\ \sin(1/x) : x > 0\} \subset \mathbb{R}^2$ and consider $A = \{(0,0)\} \cup \Gamma$. Note that $(0,0)$ is in the closure $\bar{\Gamma}$ of Γ. Then A is connected but it is not path connected: there is no continuous function inside A joining $\{(0,0)\}$ with a point of Γ.

3. Semialgebraic compactness

Definition 3.1. A semialgebraic set $A \subset R^n$ is **semialgebraic compact** if for every semialgebraic path $\alpha\colon]0,1[\longrightarrow A$,

$$\exists \lim_{t \to 0^+} \alpha(t) \in A.$$

Theorem 3.2. *Let $A \subseteq R^n$ be a semialgebraic set. Then*

$$A \text{ is semialgebraic compact} \iff A \text{ is closed and bounded.}$$

Proof.

($\Leftarrow$) Let $A \subseteq R^n$ be closed and bounded and $\alpha\colon]0,1[\to A$ a semialgebraic path.

Since A is bounded, α can be continuously extended to 0, so

$$\exists \lim_{t \to 0^+} \alpha(t) = x \in R^n$$

and $x = \alpha(0)$.

But A is closed, then $\alpha(0) \in A$.

($\Rightarrow$) Assume A is semialgebraic compact and suppose for a contradiction that A is not closed.

Let $x \in \bar{A}$, $x \notin A$. By the Curve Selection Lemma there is a semialgebraic continuous function $f\colon [0,1] \to R^n$ such that $f(]0,1]) \subset A$ and $f(0) = x$. Therefore

$$x = \lim_{t \to 0^+} f(t),$$

and $x \in A$, since A is semialgebraic compact. Contradiction.

To show that A is bounded we use the following corollary to the Curve Selection Lemma:

Corollary 3.3. *Let $A \subseteq R^n$ be an unbounded semialgebraic set. Then there is a semialgebraic path $\alpha\colon]0,1[\to A$ with*

$$\lim_{t \to 0} |\alpha(t)| = \infty.$$

□

The following Theorem and its Corollory is a particular indication that the notion of "semialgebraic compactness" is the correct analogue to usual compactness, adapted to the semialgebraic setting:

Theorem 3.4. *Let A, B semialgebraic sets and $f\colon A \to B$ a semialgebraic continuous map. Then*

$$A \text{ semialgebraic compact} \;\Rightarrow\; f(A) \text{ semialgebraic compact}.$$

Proof. We assume the following Lemma:

Lemma 3.5. *Let $f\colon A \to B$ be a semialgebraic map with A, B semialgebraic sets. Let $\beta\colon]0,1[\to B$ be a semialgebraic path in B with $\beta(]0,1[) \subseteq f(A)$. Then there is $0 < c \leqslant 1$ and a semialgebraic continuous function $\alpha\colon]0,c[\to A$ such that $\beta(t) = f(\alpha(t))$ for every $0 < t < c$.*

Let $\beta\colon]0,1[\to f(A)$ be a semialgebraic path. We want to show that

$$\exists \lim_{t \to 0^+} \beta(t) \in f(A).$$

By Lemma 3.5, there is $0 < c \leqslant 1$ and a semialgebraic continuous function $\alpha\colon]0,c[\to A$ such that $\beta(t) = f(\alpha(t))$ for every $0 < t < c$. Since A is semialgebraic compact

$$\exists \lim_{t \to 0^+} \alpha(t) = x \in A.$$

So $\lim_{t \to 0^+} \beta(t) = f(x) \in f(A)$, as required. □

Corollary 3.6. *If A is a semialgebraic compact set then any semialgebraic continuous function $f\colon A \to R$ takes maximum and minimum.*

Proof. By Thereom above $f(A)$ is semialgebraic compact, so by 3.2 it is closed and bounded. So $f(A)$ is a union of finitely many intervals $[a_i, b_i]$ (with $a_i \leqslant b_i \in R$). □