

REAL ALGEBRAIC GEOMETRY LECTURE NOTES
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1. REAL CLOSED FIELDS

We first recall Artin-Schreier characterization of real closed fields:

Proposition 1.1. (*Artin-Schreier, 1926*) *Let K be a field. The following are equivalent:*

- (i) K is real closed.
- (ii) K has an ordering P which does not extend to any proper algebraic extension.
- (iii) K is real, has no proper algebraic extension of odd degree, and

$$K = K^2 \cup -(K^2).$$

Corollary 1.2. *If K is a real closed field then*

$$K^2 = \{a^2 : a \in K\}$$

is the unique ordering of K .

Proof. Since K is a real closed field, by (ii) it has an ordering P which does not extend to any proper algebraic extension.

Let $b \in P$. Then $b = a^2$ for some $a \in K$, otherwise P extends to an ordering of $K(\sqrt{b})$, which is a proper algebraic extension of K .

Therefore $P = K^2$. □

Remark 1.3. We denote by $\sum K^2$ the unique ordering of a real closed field K , even though we know that $\sum K^2 = K^2$, to avoid any confusion with the cartesian product $K \times K$.

Corollary 1.4. *Let (K, \leq) be an ordered field. Then K is real closed if and only if*

- (a) every positive element in K has a square root in K , and
- (b) every polynomial of odd degree has a root in K .

Examples 1.5. \mathbb{R} is real closed and \mathbb{Q} is not.

2. THE ALGEBRAIC CLOSURE OF A REAL CLOSED FIELD

Lemma 2.1. (*Hilfslemma*) *If K is a field such that K^2 is an ordering of K , then every element of $K(\sqrt{-1})$ is a square.*

Proof. Let $x = a + \sqrt{-1}b \in K(\sqrt{-1}) := L$, $a, b \in K$, $b \neq 0$. We can suppose $b > 0$. We want to find $y \in L$ such that $x = y^2$.

K^2 is an ordering $\Rightarrow a^2 + b^2 \in K^2$. Let $c \in K$, $c \geq 0$ such that

$$a^2 + b^2 = c^2.$$

Since $a^2 \leq a^2 + b^2 = c^2$, $|a| \leq c$, so $c + a \geq 0$, $c - a \geq 0$ ($-c \leq a \leq c$).
Therefore $\frac{1}{2}(c \pm a) \in K^2$. Let $d, e \in K$, $d, e \geq 0$ such that

$$\frac{1}{2}(c + a) = d^2$$

$$\frac{1}{2}(c - a) = e^2.$$

So

$$d = \frac{\sqrt{c+a}}{\sqrt{2}} \quad e = \frac{\sqrt{c-a}}{\sqrt{2}}$$

Now set $y := d + e\sqrt{-1}$. Then

$$\begin{aligned} y^2 &= (d + e\sqrt{-1})^2 \\ &= d^2 + (e\sqrt{-1})^2 + 2de\sqrt{-1} \\ &= \frac{1}{2}(c+a) - \frac{1}{2}(c-a) + 2\frac{1}{2}\sqrt{(c-a)(c+a)}\sqrt{-1} \\ &= \frac{1}{2}a + \frac{1}{2}a + \sqrt{c^2 - a^2}\sqrt{-1} \\ &= a + \sqrt{b^2}\sqrt{-1} \\ &= a + b\sqrt{-1} \\ &= x. \end{aligned}$$

□

Theorem 2.2. (*Fundamental Theorem of Algebra*) *If K is a real closed field then $K(\sqrt{-1})$ is algebraically closed.*

Proof. Let $L \supseteq K(\sqrt{-1})$ be an algebraic extension of $K(\sqrt{-1})$. We show $L = K(\sqrt{-1})$.

Set $G := \text{Gal}(L/K)$. Then $[L : K] = |G| = 2^a m$, $a \geq 1$, m odd.

Let $S < G$ be a 2-Sylow subgroup ($|S| = 2^a$), and $F := \text{Fix}(S)$. We have

$$[F : K] = [G : S] = m \quad \text{odd.}$$

Since K is real closed, it follows that $m = 1$, so $G = S$ and $|G| = 2^a$. Now

$$[L : K(\sqrt{-1})][K(\sqrt{-1}) : K] = [L : K] = 2^a.$$

Therefore $[L : K(\sqrt{-1})] = 2^{a-1}$. We claim that $a = 1$.

If not, set $G_1 := \text{Gal}(L/K(\sqrt{-1}))$, let S_1 be a subgroup of G_1 of index 2, and $F_1 := \text{Fix}(S_1)$. So

$$[F_1 : K(\sqrt{-1})] = [G_1 : S_1] = 2,$$

and F_1 is a quadratic extension of $K(\sqrt{-1})$. But every element of $K(\sqrt{-1})$ is a square by Lemma 2.1, contradiction. \square

Notation. We denote by \bar{K} the algebraic closure of a field K , i.e. the smallest algebraically closed field containing K .

We have just proved that if K is real closed then $\bar{K} = K(\sqrt{-1})$.

3. FACTORIZATION IN $R[x]$

Corollary 3.1. *(Irreducible elements in $R[x]$ and prime factorization in $R[x]$). Let R be a real closed field, $f(x) \in R[x]$. Then*

(1) *if $f(x)$ is monic and irreducible then*

$$f(x) = x - a \quad \text{or} \quad f(x) = (x - a)^2 + b^2, \quad b \neq 0;$$

(2)

$$f(x) = d \prod_{i=1}^n (x - a_i) \prod_{j=1}^m (x - d_j)^2 + b_j^2, \quad b_j \neq 0.$$

Proof. Let $f(x) \in R[x]$ be monic and irreducible. Then $\deg(f) \leq 2$.

Suppose not, and let $\alpha \in \bar{R}$ a root of $f(x)$. Then

$$[R(\alpha) : R] = \deg(f) > 2.$$

On the other hand, by 2.2

$$[R(\alpha) : R] \leq [\bar{R} : R] = 2,$$

contradiction.

If $\deg(f) = 1$, then $f(x) = x - a$, for some $a \in R$.

If $\deg(f) = 2$, then $f(x) = x^2 - 2ax + c = (x - a)^2 + (c - a^2)$, for some $a, c \in R$.

We claim that $c - a^2 > 0$. If not,

$$c - a^2 \leq 0 \Rightarrow -(c - a^2) \geq 0 \Rightarrow a^2 - c \geq 0,$$

the discriminant $4(a^2 - c) \geq 0$, $f(x)$ has a root in R and factors, contradiction.

Therefore $(c - a^2) \in R^2$ and there is $b \in R$ such that $(c - a^2) = b^2 \neq 0$. \square

Corollary 3.2. (*Zwischenwertsatz : Intermediate value Theorem*) Let R be a real closed field, $f(x) \in R[x]$. Assume $a < b \in R$ with $f(a) < 0 < f(b)$. Then $\exists c \in R$, $a < c < b$ such that $f(c) = 0$.

Proof. By previous Corollary,

$$\begin{aligned} f(x) &= d \prod_{i=1}^n (x - a_i) \prod_{j=1}^m (x - d_j)^2 + b_j^2 \\ &= d \prod_{i=1}^n l_i(x) q(x), \end{aligned}$$

where $l_i(x) := x - a_i$, $\forall i = 1, \dots, n$ and $q(x) := \prod_{j=1}^m (x - d_j)^2 + b_j^2$.

We claim that there is some $k \in \{1, \dots, n\}$ such that $l_k(a)l_k(b) < 0$. Since

$$\text{sign}(f) = \text{sign}(d) \prod_{i=1}^n \text{sign}(l_i) \text{sign}(q) \quad \text{and} \quad \text{sign}(q) = 1,$$

if we had that

$$\text{sign}(l_i(a)) = \text{sign}(l_i(b)) \quad \forall i \in \{1, \dots, n\},$$

we would have

$$\text{sign}(f(a)) = \text{sign}(f(b)),$$

in contradiction with $f(a)f(b) < 0$.

For such a k ,

$$l_k(a) < 0 < l_k(b),$$

i.e.

$$a - a_k < 0 < b - a_k,$$

and $c := a_k \in]a, b[$ is a root of $f(x)$. □

Corollary 3.3. (*Rolle*) Let R be a real closed field, $f(x) \in R[x]$, Assume that $a, b \in R$, $a < b$ and $f(a) = f(b) = 0$. Then $\exists c \in R$, $a < c < b$ such that $f'(c) = 0$.