

REAL ALGEBRAIC GEOMETRY LECTURE NOTES
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Convention: When a new definition is given, the German name appears between brackets.

1. ORDERINGS

Definition 1.1. (*partielle Anordnung*) Let Γ be a non-empty set and let \leq be a relation on Γ such that:

$$(i) \quad \gamma \leq \gamma \quad \forall \gamma \in \Gamma,$$

$$(ii) \quad \gamma_1 \leq \gamma_2, \gamma_2 \leq \gamma_1 \Rightarrow \gamma_1 = \gamma_2 \quad \forall \gamma_1, \gamma_2 \in \Gamma,$$

$$(iii) \quad \gamma_1 \leq \gamma_2, \gamma_2 \leq \gamma_3 \Rightarrow \gamma_1 \leq \gamma_3 \quad \forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma.$$

Then \leq is a **partial order** on Γ and (Γ, \leq) is said to be a **partially ordered set**.

Example 1.2. Let X be a non-empty set. For every $A, B \subseteq X$, the relation

$$A \leq B \iff A \subseteq B,$$

is a partial order on the power set $\mathcal{P}(X) = \{A : A \subseteq X\}$.

Definition 1.3. (*totale Anordnung*) A partial order \leq on a set Γ is said to be **total** if

$$\forall \gamma_1, \gamma_2 \in \Gamma \quad \gamma_1 \leq \gamma_2 \text{ or } \gamma_2 \leq \gamma_1.$$

Notation 1.4. If (Γ, \leq) is a partially ordered set and $\gamma_1, \gamma_2 \in \Gamma$, then we write:

$$\gamma_1 < \gamma_2 \iff \gamma_1 \leq \gamma_2 \text{ and } \gamma_1 \neq \gamma_2,$$

$$\gamma_1 \geq \gamma_2 \iff \gamma_2 \leq \gamma_1,$$

$$\gamma_1 > \gamma_2 \iff \gamma_2 \leq \gamma_1 \text{ and } \gamma_1 \neq \gamma_2.$$

Examples 1.5. Let $\Gamma = \mathbb{R} \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}\}$.

(1) For every $(a_1, b_1), (a_2, b_2) \in \mathbb{R} \times \mathbb{R}$ we can define

$$(a_1, b_1) \leq (a_2, b_2) \iff a_1 \leq a_2 \text{ and } b_1 \leq b_2.$$

Then $(\mathbb{R} \times \mathbb{R}, \leq)$ is a partially ordered set.

(2) For every $(a_1, b_1), (a_2, b_2) \in \mathbb{R} \times \mathbb{R}$ we can define

$$(a_1, b_1) \leq_l (a_2, b_2) \iff [a_1 < a_2] \text{ or } [a_1 = a_2 \text{ and } b_1 \leq b_2].$$

Then $(\mathbb{R} \times \mathbb{R}, \leq_l)$ is a totally ordered set. (Remark: the "l" stands for "lexicographic").

2. ORDERED FIELDS

Definition 2.1. (*angeordneter Körper*) Let K be a field. Let \leq be a total order on K such that:

$$(i) \quad x \leq y \Rightarrow x + z \leq y + z \quad \forall x, y, z \in K,$$

$$(ii) \quad 0 \leq x, 0 \leq y \Rightarrow 0 \leq xy \quad \forall x, y \in K.$$

Then the pair (K, \leq) is said to be an **ordered field**.

Examples 2.2. The field of the rational numbers (\mathbb{Q}, \leq) and the field of the real numbers (\mathbb{R}, \leq) are ordered fields, where \leq denotes the usual order.

Definition 2.3. (*formal reell Körper*) A field K is said to be **(formal) real** if there is an order \leq on K such that (K, \leq) is an ordered field.

Proposition 2.4. Let (K, \leq) be an ordered field. The following hold:

- $a \leq b \iff 0 \leq b - a \quad \forall a, b \in K$
- $0 \leq a^2 \quad \forall a \in K$
- $a \leq b, 0 \leq c \Rightarrow ac \leq bc \quad \forall a, b, c \in K$
- $0 < a \leq b \Rightarrow 0 < 1/b \leq 1/a \quad \forall a, b \in K$
- $0 < n \quad \forall n \in \mathbb{N}$

Remark 2.5. If K is a real field then $\text{char}(K) = 0$ and K contains a copy of \mathbb{Q} .

Notation 2.6. Let (K, \leq) be an ordered field and let $a \in K$.

$$\text{sign}(a) := \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

$$|a| := \text{sign}(a)a.$$

Fact 2.7. Let (K, \leq) be an ordered field and let $a, b \in K$. Then

$$(i) \text{ sign}(ab) = \text{sign}(a) \text{sign}(b),$$

$$(ii) |ab| = |a||b|,$$

$$(iii) |a + b| \leq |a| + |b|.$$

3. ARCHIMEDEAN FIELDS

Definition 3.1. (*archimedischer Körper*) Let (K, \leq) be a field. We say that K is **Archimedean** if

$$\forall a \in K \exists n \in \mathbb{N} \text{ such that } a < n.$$

Definition 3.2. Let (Γ, \leq) be an ordered set and let $\Delta \subseteq \Gamma$. Then

- Δ is **cofinal** (*kofinal*) in Γ if

$$\forall \gamma \in \Gamma \exists \delta \in \Delta \text{ such that } \gamma \leq \delta.$$

- Δ is **coinitial** (*koinitial*) in Γ if

$$\forall \gamma \in \Gamma \exists \delta \in \Delta \text{ such that } \delta \leq \gamma.$$

- Δ is **coterminal** (*koterminal*) in Γ if Δ is cofinal and coinitial in Γ .

Example 3.3. Let (K, \leq) be an Archimedean field. Then \mathbb{N} is cofinal in K , $-\mathbb{N}$ is coinitial in K and $\mathbb{Z} = -\mathbb{N} \cup \mathbb{N}$ is coterminal in K .

Remark 3.4.

- If (K, \leq) is an Archimedean field and $Q \subseteq K$ is a subfield, then (Q, \leq) is an Archimedean field.
- (\mathbb{R}, \leq) is an Archimedean field and therefore also (\mathbb{Q}, \leq) is.

Remark 3.5. Let (K, \leq) be an ordered field. Then K is Archimedean if and only if $\forall a, b \in K^* \exists n \in \mathbb{N}$ such that

$$|a| \leq n|b| \text{ and } |b| \leq n|a|.$$

Example 3.6. Let $\mathbb{R}[x]$ be the ring of the polynomials with coefficients in \mathbb{R} . We denote by $ff(\mathbb{R}[x])$ the field of the rational functions of $\mathbb{R}[x]$, i.e.

$$ff(\mathbb{R}[x]) = \mathbb{R}(x) := \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{R}[x] \text{ and } g(x) \neq 0 \right\}.$$

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ and let $k \in \mathbb{N}$ the smallest index such that $a_k \neq 0$ (and therefore actually $f(x) = a_n x^n + \cdots + a_k x^k$). We define

$$f(x) > 0 \Leftrightarrow a_k > 0$$

and then for every $f(x), g(x) \in \mathbb{R}[x]$ with $g(x) \neq 0$ we define

$$\frac{f(x)}{g(x)} \geq 0 \Leftrightarrow f(x)g(x) \geq 0.$$

This is a total order on $K = f f(\mathbb{R}[x])$ which makes (K, \leq) an ordered field. We claim that (K, \leq) contains

(i) an infinite positive element, i.e.

$$\exists A \in K \text{ such that } A > n \quad \forall n \in \mathbb{N},$$

(ii) an infinitesimal positive element, i.e.

$$\exists a \in K \text{ such that } 0 < a < 1/n \quad \forall n \in \mathbb{N}.$$

For instance the element $x \in K$ is infinitesimal and the element $1/x \in K$ is infinite. Therefore (K, \leq) is not Archimedean.