Universität Konstanz

Existence and regularity of random attractors for evolution equations with rough noise

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joint work with Alexandra Blessing (Neamțu)

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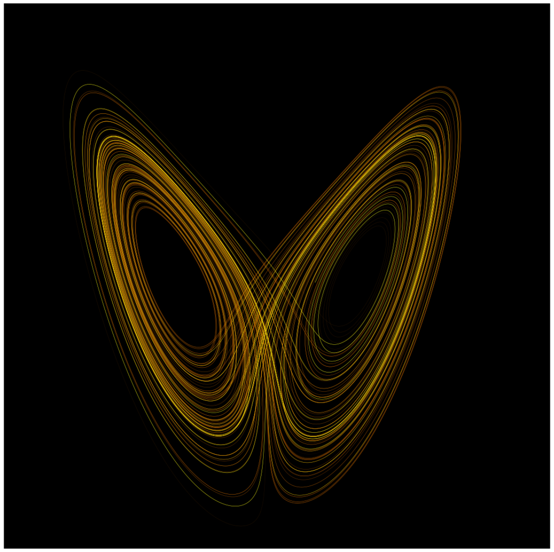
Outline

Motivational example

- Most dynamical systems arise from equations derived from physical applications
  - $\rightarrow$ Navier–Stokes equations, Lorentz-63 model, climate models,...

Lorentz-63 model:

$$\begin{cases} \mathrm{d}x = (s(y-x))\,\mathrm{d}t \\ \mathrm{d}y = (rx - y - xz)\,\mathrm{d}t \\ \mathrm{d}z = (-bz + xy)\,\mathrm{d}t \end{cases}$$



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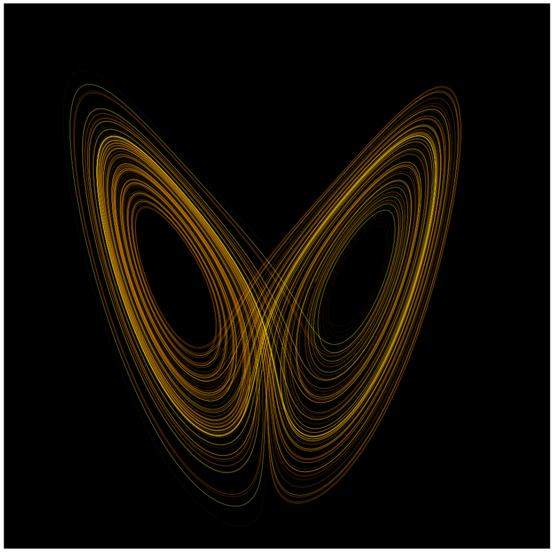


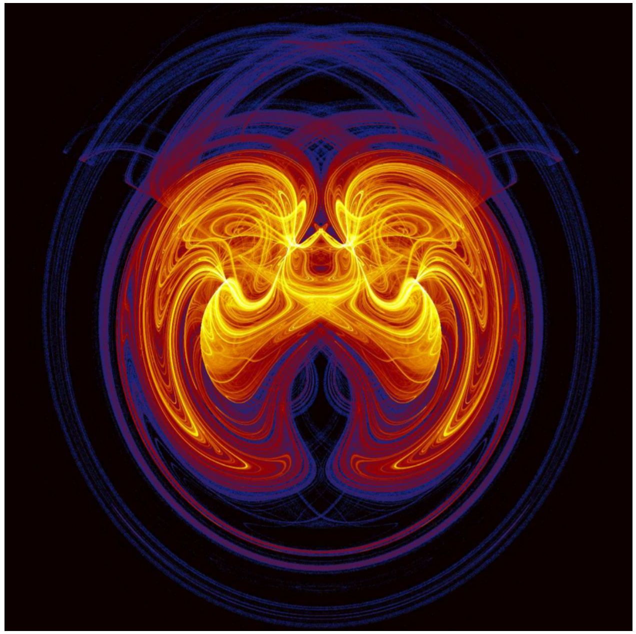
Why are we interested in attractors?

$\Rightarrow$ An attractor describes the long-time behavior of the system!

How to treat the randomness?

Stochastic Lorentz-63 model with a Brownian motion $(W_t)_{t\in\mathbb{R}}$

$$\begin{cases} \mathrm{d}x = (s(y-x))\,\mathrm{d}t + \sigma x\,\mathrm{d}W_t \\ \mathrm{d}y = (rx - y - xz)\,\mathrm{d}t + \sigma y\,\mathrm{d}W_t \\ \mathrm{d}z = (-bz + xy)\,\mathrm{d}t + \sigma z\,\mathrm{d}W_t \end{cases}$$



Noise can be seen as **non-autonomous** forcing!

We need to define a suitable concept of a random attractor.
$\Rightarrow$ We use results from the non-autonomous theory.

Chekroun-Simonett-Ghil, *Stochastic climate dynamics: Random attractors and time-dependent invariant measures*, 2010.

Autonomous case

$$\dot{y}_t = -y_t, y(t_0) = y_0 \in \mathbb{R}$$

has the global solution

$$y(t, t_0; y_0) = y_0 e^{-(t-t_0)}.$$

Forward dynamics

$$\lim_{t \to \infty} y(t, t_0; y_0) = 0$$

Backward dynamics

$$\lim_{t_0 \to -\infty} y(t, t_0; y_0) = 0$$

Remark

Forward and backward dynamics are the same, since the evolution only depends on the time that has passed.

Non-autonomous case

$$\dot{y}_t = -y_t + t, y(t_0) = y_0 \in \mathbb{R}$$

has the global solution

$$y(t, t_0; y_0) = t - 1 + (y_0 - t_0 + 1)\, e^{-(t-t_0)}.$$

Forward dynamics

$$\lim_{t\to\infty} y(t, t_0; y_0) = \infty$$

Backward dynamics

$$\lim_{t_0\to-\infty} y(t, t_0; y_0) = t - 1$$

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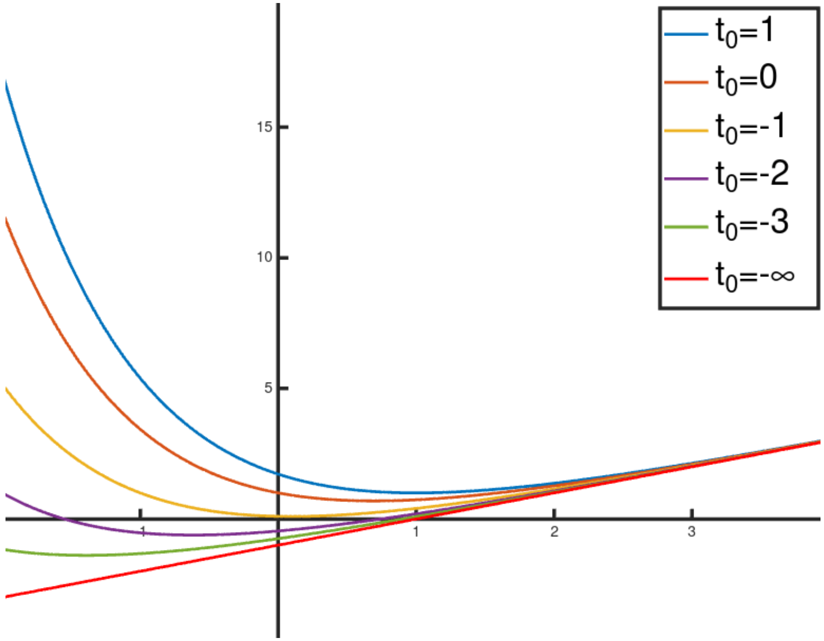

Stochastic case

Consider the canonical probability space

$$(\Omega, \Sigma, \mathbb{P}) := (C_0(\mathbb{R}; \mathbb{R}), \mathcal{B}(\Omega), \mathbb{P}_W)$$

such that $W_t(\omega) := \omega_t$ is a two-sided Brownian motion. We introduce the **Wiener shift**

$$\theta : \mathbb{R} \times \Omega \to \Omega, (t, \omega) \mapsto \theta_t \omega := \omega(t + \cdot) - \omega(t),$$

which models the time evolution of the noise.
The equation

$$\mathrm{d}y_t(\omega) = -y_t(\omega)\,\mathrm{d}t + \mathrm{d}W_t(\omega), y_0 \in \mathbb{R}$$

has the global solution

$$y(t, \omega; y_0) = y_0 e^{-t} + \int_0^t e^{-(t-r)}\,\mathrm{d}W_r(\omega) = y_0 e^{-t} + \int_{-t}^0 e^r \mathrm{d}\theta_t W_r(\omega).$$

In this case

$$\lim_{t \to \infty} y(t, \theta_{-t}\omega; y_0) = \int_{-\infty}^0 e^r\,\mathrm{d}W_r(\omega) =: \mathcal{A}(\omega).$$

Random dynamical systems

Definition (Metric dynamical system)

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and define the measurable function $\theta : \mathbb{R} \times \Omega \to \Omega, (t, \omega) \mapsto \theta_t\omega$ such that $\mathbb{P} \circ \theta_t^{-1} = \mathbb{P}$, $\theta_0 = \mathrm{Id}_\Omega$ and

$$\theta_{t+s} = \theta_t \circ \theta_s \quad \text{for all} \quad t, s \in \mathbb{R}.$$

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Definition (Random dynamical system)

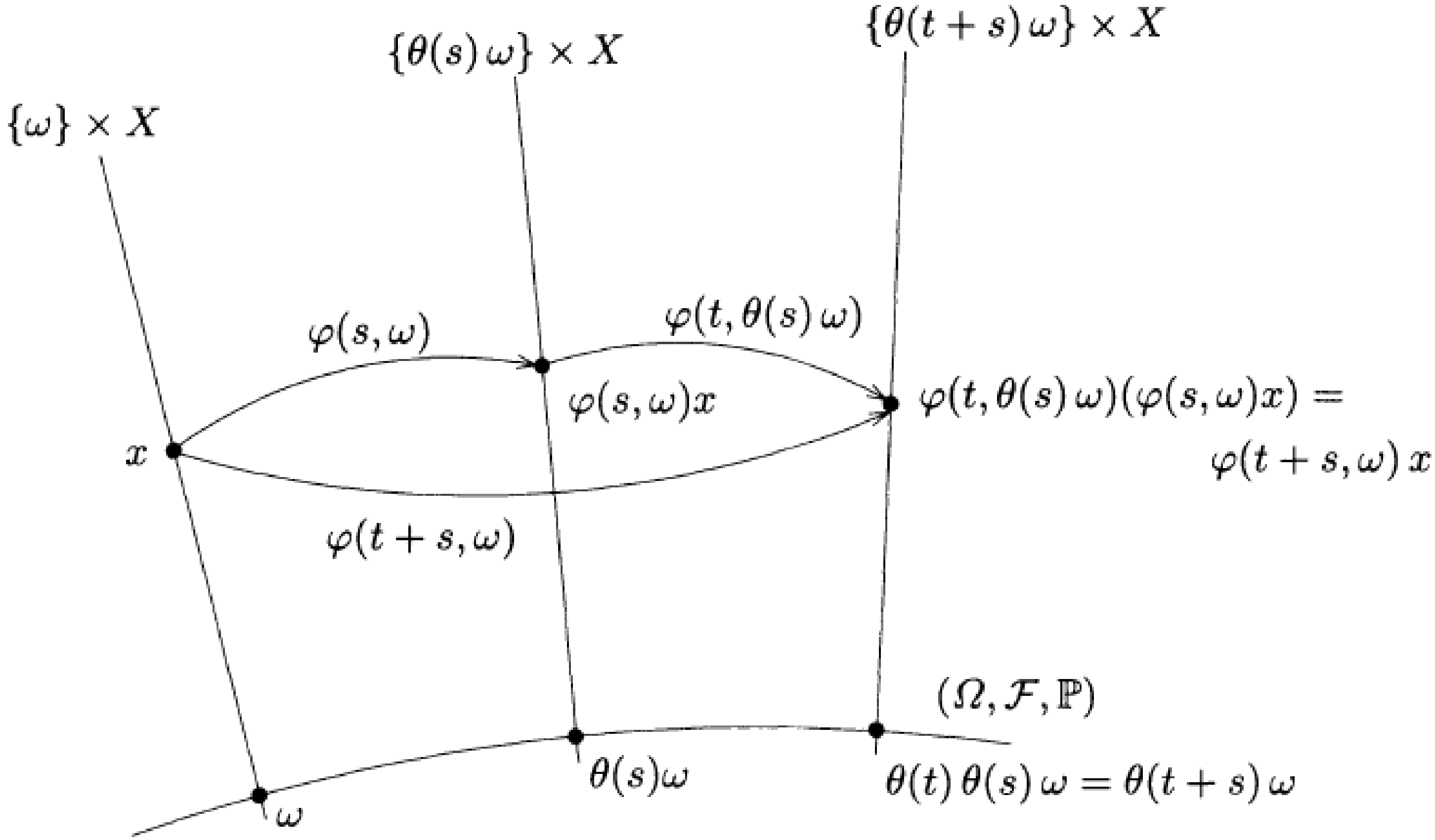
Given a MDS $(\Omega, \Sigma, \mathbb{P}, \theta)$ we call a measurable function $\phi : \mathbb{R} \times \Omega \times E \to E$ a random dynamical system if the following properties are fulfilled:

- $\phi(0, \omega, x) = x$ for all $\omega \in \Omega, x \in E$,
- for all $t, s \in \mathbb{R}, \omega \in \Omega$ and $x \in E$ we have the cocycle property

$$\phi(t + s, \omega, x) = \phi(t, \theta_s \omega, \phi(s, \omega, x)),$$

- $\phi(t, \omega, \cdot)$ is continuous for all $\omega \in \Omega$ and $t \in \mathbb{R}$.

The cocycle property



Random attractor

We call a positive random variable R tempered if for all $\beta > 0$

$$\lim_{t \to \pm\infty} e^{-\beta|t|} R(\theta_t \omega) = 0.$$

$(B(\omega))_{\omega \in \Omega}$ is tempered if $R(\omega) := \sup_{x \in B(\omega)} \|x\|$ is tempered.

Definition (Crauel-Flandoli; 1994)

We call a random set $(\mathcal{A}(\omega))_{\omega \in \Omega}$ a global random (pullback) attractor associated to the random dynamical system ϕ if $\emptyset \neq \mathcal{A}(\omega) \subset E$, $\mathcal{A}(\omega)$ is compact, $\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t \omega)$ and $\mathcal{A}(\omega)$ pullback attracts all tempered random sets $(B(\omega))_{\omega \in \Omega}$, that means

$$\lim_{t \to \infty} \text{dist}_H(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0.$$

- Different notions of attractors possible
- How to obtain an RDS from a SPDE?

Crauel-Flandoli, *Attractors for random dynamical systems*, Probab. Theory Related Fields, 100(3):365-393, 1994

Why rough paths?

$$\mathrm{d}y_t = (Ay_t + F(y_t))\,\mathrm{d}t + \sigma \mathrm{d}W_t \tag{1}$$

Set $\phi(t, \omega, \cdot) : E \to E$ the solution map. The cocycle property

$$\phi(t+s, \omega, x) = \phi(t, \theta_s\omega, \phi(s, \omega, x)),$$

needs to hold **for all** $\omega \in \Omega$! In infinite dimensions not possible.

Imkeller-Schmalfuss, *The Conjugacy of Stochastic and Random Differential Equations and the Existence of Global Attractors*, J. Dynam. Differential Equations, 13(2):215–249, 2001.

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Instead: Look at the transformed **random PDE**

$$\dot{v}_t = Av_t + F(v_t + Z_t) \tag{2}$$

where $y = v + Z$ and Z is the stationary solution of

$$\mathrm{d}Z_t = AZ_t\,\mathrm{d}t + \sigma \mathrm{d}W_t.$$

$\Rightarrow$ (2) can be solved **pathwise**!

Idea: Find attractor of (2) and „transform“ back to get an attractor of (1).

Does not work for general multiplicative noise! $\Rightarrow$ Work with rough paths, which directly gives a pathwise solution.

Imkeller-Schmalfuss, *The Conjugacy of Stochastic and Random Differential Equations and the Existence of Global Attractors*, J. Dynam. Differential Equations, 13(2):215–249, 2001.

Quick overview of rough paths

Definition (Hölder rough path)

Let $X \in C^\gamma$ be a path for $\frac{1}{3} < \gamma \leq \frac{1}{2}$ such that some two-parameter function $\mathbb{X} \in C_2^{2\gamma}$ exists such that

$$X_{s,t} - X_{s,u} - X_{u,t} = \mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t}$$

for all $s \leq u \leq t$. The $\mathbf{X} := (X, \mathbb{X})$ is a γ-Hölder rough path.

Let $(W_t^H)_{t\in\mathbb{R}}$ be a Gaussian process such that

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

For $H \in (\frac{1}{3}, \frac{1}{2})$ there exists $\mathbb{W}^H$ such that $(W^H(\omega), \mathbb{W}^H(\omega))$ is a rough path for all ω.

Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoam., 14(2):215–310, 1998.
Friz-Hairer, *A Course on Rough Paths, with an introduction to regularity structures*, Springer, 2014.

Quick overview of rough paths

Let A be the generator of an analytic semigroup $(S(t))_{t\geq 0}$ on E and define the fractional powers

$$E_\alpha = D(A^\alpha),$$

for $\alpha > 0$ with $\|\cdot\|_\alpha := \|A^\alpha \cdot\|$.

Definition (Controlled rough path)

A pair $(y, y') \in C(E_\alpha) \times ((C(E_{\alpha-\gamma}) \cap C^\gamma(E_{\alpha-2\gamma}))$ is called a controlled rough path if there exists a remainder $R^y \in C^\gamma(E_{\alpha-\gamma}) \cap C^{2\gamma}(E_{\alpha-2\gamma})$ such that

$$y_t = y_s + y'_s X_{s,t} + R^y_{s,t}.$$

Gubinelli-Tindel, *Rough evolution equations*, Ann Probab., 38(1):1–75, 2010.
Gerasimovičs-Hocquet-Nilssen, *Non-autonomous rough semilinear PDEs and the multiplicative Sewing lemma*, J. Funct. Anal., 281(10):65, 2021.

Integration against rough paths

Theorem (Integration)

Let $\mathbf{X}$ is γ-Hölder rough path with $\frac{1}{3} < \gamma \leq \frac{1}{2}$, A generating an analytic semigroup $(S(t))_{t\geq 0}$ on E, $G : E_{\alpha-i\gamma} \to E_{\alpha-i\gamma-\sigma}$ is 3-times Fréchet differentiable with bounded derivatives for $i = 0, 1, 2$ and some $\sigma \in [0, \gamma)$. Then

$$\int_s^t S(t-r)G(y_r)\,\mathrm{d}\mathbf{X}_r$$

exists, where (y, y') is a controlled rough path.

Gubinelli-Tindel, *Rough evolution equations*, Ann Probab., 38(1):1–75, 2010.

Gerasimovičs-Hocquet-Nilssen, *Non-autonomous rough semilinear PDEs and the multiplicative Sewing lemma*, J. Funct. Anal., 281(10):65, 2021.

Setting

$$dy_t = Ay_t\,dt + G(y_t)\,d\mathbf{X}_t,\ y_0 \in E \tag{3}$$

Known results for equation (3)

- Local solution [Gerasimovičs-Hairer; 2019], [Gubinelli-Tindel; 2010], [Gerasimovičs-Hocquet-Nilssen; 2021]
- Global solution [Hesse-Neamțu; 2022]
- Existence of center manifolds [Kuehn-Neamțu; 2021], [Kühn-Neamțu; 2023],[Riedel-Varzaneh; 2023]
- Exponential stability [Hesse; 2022]
- Attractor for rough PDEs [Yang-Lin-Zeng; 2023]

Theorem (Hesse-Neamțu; 2022)

Suppose that A generates an analytic semigroup $(S(t))_{t\geq 0}$, $G : E_{\alpha-i\gamma} \to E_{\alpha-i\gamma-\sigma}$ is bounded and 3-times Fréchet differentiable with bounded derivatives for $i = 0, 1, 2$ and some $\sigma \in [0, \gamma)$ and $\mathbf{X}$ is an enhanced fBM with $\frac{1}{3} < H \leq \frac{1}{2}$. Then the solution of (3) forms a RDS ϕ.

Hesse-Neamțu, *Global solutions for semilinear rough partial differential equations*, Stoch. Dyn., 22(2):1–18, 2022.

Existence of random attractors

$$dy_t = Ay_t \, dt + G(y_t) \, d\mathbf{X}_t, \, y_0 \in E \tag{3}$$

Theorem (Flandoli-Schmalfuss; 1996)

Let $(K(\omega))_{\omega \in \Omega}$ be compact, tempered and absorbing, that means for every tempered set $(D(\omega))_{\omega \in \Omega}$ there is a $T_D > 0$ with

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset K(\omega) \qquad \text{for all} \quad t \geq T_D.$$

Then (3) has an unique random pullback attractor $(\mathcal{A}(\omega))_{\omega \in \Omega}$ given by

$$\mathcal{A}(\omega) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))}.$$

Flandoli-Schmalfuss, *Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise*, Stoch. Stoch. Rep., 51(1-2):21-45, 1996.

Main result

Theorem (Neamțu-S. 2024)

The RPDE has a global random attractor, provided that $y_0 \in E_\alpha$ and that the condition $\lambda_A \geq C(G, S, \mathbf{X})$, where $\|S(t)\|_{\mathcal{L}(E)} \lesssim e^{-\lambda_A t}$, is satisfied.

$\Rightarrow$ Improvement to [Yang-Lin-Zeng; 2023], where G is assumed to be small.

Step 0: Prove global-in-time existence

$$y_t := S(t)y_0 + \int_0^t S(t-r)G(y_r)\,\mathrm{d}\mathbf{X}_r.$$

Step 1: Integrable bound for the solution on bounded intervals $[s, t]$

$$\|y_r\|_\alpha \leq \|y_s\|_\alpha P_1(\omega, [s,t]) + P_2(\omega, [s,t]).$$

Step 2: Discretize the rough integral.

Riedel-Varzaneh, *An integrable bound for rough stochastic partial differential equations with applications to invariant manifolds and stability*, arxiv.org/abs/2307.01679, 2023.

Cass-Litterer-Lyons, *Integrability and tail estimates for Gaussian rough differential equations*, Ann. Probab., 41(4):3026–3050, 2013.

Main result

Step 3: Estimate y_t for $t \in [n, n+1]$

$$\|y_t\|_\alpha e^{\lambda t} \lesssim C(t, y_0) + \sum_{l=0}^{n} e^{\lambda l} P(\omega, [l, l+1]).$$

Step 4: Estimate the solution on discrete times $y_n \Rightarrow$ Right-hand side no longer depends on the solution

Step 5: Estimate the RDS for $t \in [n, n+1]$

$$\begin{aligned}\|\phi(t, \theta_{-t}\omega, y_0(\theta_{-t}\omega))\|_\alpha &\leq \|y_n\|_\alpha P_1(\theta_{-t}\omega, [n, n+1]) + P_2(\theta_{-t}\omega, [n, n+1]) \\ &\leq R(\omega)\end{aligned}$$

Step 6: Prove temperedness of $R(\omega)$, then $B(\omega) := B_0(R(\omega) + \varepsilon)$ is an absorbing set.

Step 7: Define

$$K(\omega) := \overline{\phi(T_B, \theta_{-T_B}\omega, B(\theta_{-T_B}\omega))}^{E_\alpha}$$

which is a compact absorbing set.

Further results

Corollary (Neamțu-S. 2024)

The attractor belongs to $E_{\alpha+\beta}$ for $\beta < \gamma$, where $X \in C^{\gamma}$.

Example: Equations with rough boundary noise [Neamțu, S.; 2023]

$$\begin{cases} \dot{y}_t = \Delta y_t & \text{in } \mathcal{O} \\ \partial_\nu y_t = G(y_t)\, \dot{\mathbf{X}}_t & \text{on } \partial\mathcal{O} \end{cases} \tag{4}$$

Corollary (Neamțu-S. 2024)

For $y_0 \in E_{-\eta}$ and $G : E_{-\eta-i\gamma} \to E_{-\eta-i\gamma+\sigma}$ bounded and 3-times bounded Fréchet differentiable, (4) has a global attractor.

Neamțu-Seitz, *Stochastic evolution equations with rough boundary noise*, Partial Differ. Equ. Appl., 4(49), 2024.

Open questions and future works

- Non-autonomous and quasilinear rough equations, ongoing work with Mazyar Varzaneh and Alexandra Blessing.
- Integrable solution bounds under the following assumptions:
  - remove the boundedness on G;
  - incorporate other noise terms (Gaussian processes).

References

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J. Funct. Anal., 281(10):Paper No. 109200, 65, 2021.

Mazyar Ghani Varzaneh and Sebastian Riedel.
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Thank you for your attention!