Stochastic Evolution Equations with Rough Boundary Noise

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Statement of the Problem

We investigate a semilinear parabolic evolution equation on a bounded domain $\mathcal{O}$ with smooth boundary and **multiplicative Neumann boundary noise** modelled by a γ-Hölder rough path $\mathbf{X} = (X, \mathbb{X})$ with $\gamma \in (1/3, 1/2]$ given by

$$\frac{\partial}{\partial t} y = \mathcal{A}y + f(y) \ \text{ in } \mathcal{O}, \quad \mathcal{C}y = F(y) \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X} \ \text{ on } \partial\mathcal{O}, \quad y(0) = y_0. \tag{1}$$

- $\mathcal{A}$ is a second order operator with Neumann boundary operator $\mathcal{C}$. A is its L^p-realization for $p \in [2,3]$, for **example** $A = \Delta$.
- We consider the mild formulation $y_t = S_t y_0 + \int_0^t S_{t-r} f(y_r)\, \mathrm{d}r + A \int_0^t S_{t-r} N(F(y_r))\, \mathrm{d}\mathbf{X}_r$ using a **controlled rough path approach**.
- In this context $N : B^{\alpha - 1/p}_{p,p}(\partial\mathcal{O}) \to H^{\alpha,p}(\mathcal{O})$ is the Neumann operator, such that $u := Ng$ is the solution of the problem $\mathcal{A}u = 0, \mathcal{C}u = g$.
- Similar problems were considered in [MP07] for additive fractional noise and [SV11] for multiplicative Brownian noise.

Controlled Rough Paths

- A **γ-Hölder rough path** is a pair $\mathbf{X} = (X, \mathbb{X})$ of some path $X \in C^\gamma([0,T];\mathbb{R})$ enhanced with $\mathbb{X} \in C^{2\gamma}([0,T]^2;\mathbb{R})$ such that Chen's relation
$$\mathbb{X}_{t,s} - \mathbb{X}_{u,s} - \mathbb{X}_{t,u} = (X_u - X_s) \otimes (X_t - X_u)$$
is satisfied.
- For a fixed path X, we say $y \in C([0,T];\mathcal{B}_\alpha)$ is a **controlled rough path** with Gubinelli derivative $y' \in C([0,T];\mathcal{B}_{\alpha-\gamma}) \cap C^\gamma([0,T];\mathcal{B}_{\alpha-2\gamma})$ if the remainder $R^y_{t,s} := y_{t,s} - y'_s X_{t,s}$ is in $C^\gamma([0,T]^2;\mathcal{B}_{\alpha-\gamma}) \cap C^{2\gamma}([0,T]^2;\mathcal{B}_{\alpha-2\gamma})$, where $(\mathcal{B}_\alpha)_{\alpha\in\mathbb{R}}$ is a monotone family of interpolation spaces. **Notation:** $(y, y') \in \mathcal{D}^{2\gamma}_{X,\alpha}$.
- **Aim.** Define the **rough convolution** as
$$\int_s^t S_{t-r} N y_r \, \mathrm{d}\mathbf{X}_r := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} S_{t-u} N y_u X_{v,u} + S_{t-u} N y'_u \mathbb{X}_{v,u} \in D(A).$$
- We consider the **Besov scale** $\widetilde{\mathcal{B}}_\alpha := B^{\alpha-1-1/p}_{p,p}(\partial\mathcal{O})$ and the **Bessel potential scale** with boundary condtions $\mathcal{B}_\beta = H^{2\beta,p}_{\mathcal{C}}(\mathcal{O})$, given by
$$H^{\beta,p}_{\mathcal{C}}(\mathcal{O}) := \begin{cases} \{u \in H^{\beta,p}(\mathcal{O}) : \mathcal{C}u = 0\}, & \beta > 1 + 1/p \\ H^{\beta,p}(\mathcal{O}), & -1 + 1/p < \beta < 1 + 1/p \end{cases}.$$
- The Neumann operator N maps into $D(A^\varepsilon) = H^{2\varepsilon,p}_{\mathcal{C}}(\mathcal{O}) = \mathcal{B}_\varepsilon$ for $\varepsilon < 1/2 + 1/2p$.
- For $(\widetilde{y}, \widetilde{y}') \in \widetilde{\mathcal{D}}^{2\gamma}_{X,\alpha}$ with $\alpha > 1 + 1/p$ we have that $(A_{-\sigma} N\widetilde{y}, A_{-\sigma} N\widetilde{y}') \in \mathcal{D}^{2\gamma}_{X,-\eta}$, where $\eta := 1 - \varepsilon$ and σ as below.
- Here $A_{-\sigma}$ is an extrapolation operator as defined below.

Why do we need extrapolation operators?

- **Problem.** The expression ANy is not well-defined, since $Ny \notin D(A)$. This is the key point, where the theory of extrapolation operators is needed.
- The **extrapolation-interpolation Banach scale** is a family $(A_\alpha, \mathcal{B}_\alpha)_{\alpha\in[-2,\infty)}$ generated by $(A, D(A))$ such that $A_\alpha \in \mathcal{L}(\mathcal{B}_{1+\alpha}, \mathcal{B}_\alpha)$, $\mathcal{B}_\alpha \hookrightarrow \mathcal{B}_\beta$ for $\alpha > \beta \geq -2$ and
$$\begin{array}{ccc} \mathcal{B}_{1+\alpha} & \hookrightarrow & \mathcal{B}_{1+\beta} \\ A_\alpha \downarrow & & \downarrow A_\beta \\ \mathcal{B}_\alpha & \hookrightarrow & \mathcal{B}_\beta \end{array}$$
is a commutative diagram.
- For negative indices, the operators are **extensions of** A and are called **extrapolated operators**.
- The index $-\sigma := -\eta - \gamma$ we choose in (2), is determined by the Neumann operator N, which maps into $D(A^\varepsilon)$, and the Hölder regularity γ of the rough path $\mathbf{X}$. We have
$$1 - \gamma < \varepsilon < 1/2 + 1/2p.$$

Main Results

- **Idea.** Rewrite (1) as a semilinear equation **without boundary noise**
$$\mathrm{d}y = (Ay + f(y))\, \mathrm{d}t + A_{-\sigma} NF(y)\, \mathrm{d}\mathbf{X}. \tag{2}$$
- For (2) we prove the existence of a **local-in-time** solution using a fixed point argument in $\mathcal{D}^{2\gamma}_{X,-\eta}$.
- Deriving estimates without quadratic terms, we also obtain a **global solution**.
- Since the construction of the solution is **pathwise**, the solution operator of (2) generates a **random dynamical system**.

Assumptions & Example

- A generates an **analytic semigroup** $(S_t)_{t\in[0,\infty)}$.
- **Example.** Second order differential operators with smooth, symmetric and uniformly elliptic coefficients
$$\mathcal{A} := \sum_{i,j} \partial_i a_{ij} \partial_j \text{ and } \mathcal{C} := \sum_{i,j} \gamma_\partial \nu_i a_{ij} \partial_j.$$
- The drift term $f : \mathcal{B}_{-\eta} \to \mathcal{B}_{-\eta-\delta_1}$ is Lipschitz and satisfies a linear growth condition. Note that f is allowed to **lose spatial regularity**.
- The diffusion coefficient $F : \mathcal{B}_{-\eta-\vartheta} \to \widetilde{\mathcal{B}}_{-\eta-\vartheta+\delta_2}$ is three times continuously differentiable with bounded derivatives for $\vartheta \in \{0, \gamma, 2\gamma\}$ and
$$DF(\cdot) \circ (A_{-\sigma} NF(\cdot))$$
has a bounded derivative. Note that F has to **gain spatial regularity**.
- **Example.** For F one can choose a lift operator
$$\Lambda^t : H^s(\mathbb{R}) \to H^{s-t}(\mathbb{R}), u \mapsto \mathcal{F}^{-1}(1 + |\cdot|^2)^{t/2} \mathcal{F}u,$$
for $t, s \in \mathbb{R}$, restricted to the bounded domain $\mathcal{O}$.

References

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