Stochastic evolution equations with rough boundary noise

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arXiv:2209.10348

TU Berlin, 26. January 2023

Outline

General Setting

We consider a semilinear evolution equation, with boundary noise where $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain with smooth boundary

$$\begin{aligned} \dot{y}_t &= \mathcal{A} y_t + f(y_t) \quad \text{in } \mathcal{O}, \\ \mathcal{C} y_t &= F(y_t)\, \dot{\mathbf{X}}_t \qquad \text{on } \partial\mathcal{O}. \end{aligned}$$

With

$$\mathcal{A} = \sum_{i,j=1}^{d} \partial_i (a_{ij} \partial_j) + a_0 \qquad \mathcal{C} = \sum_{i,j=1}^{d} \gamma_\partial a_{ij} \nu_i \partial_j,$$

and $A : D(A) \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is the realization according to this boundary value problem.

General idea

For e.g. the heat equation and additive fractional Brownian motion with $H \in (0,1)$

$$\begin{aligned} \dot{y}_t &= \Delta y_t \quad \text{in } \mathcal{O}, \\ \mathcal{C} y_t &= \Phi \ \dot{B}^H_t \quad \text{on } \partial\mathcal{O}. \end{aligned} \tag{1}$$

- The solution operator N of the elliptic boundary value problem $\Delta u = 0,\ \mathcal{C}u = g$ maps $L^2(\partial\mathcal{O})$ into $D(\Delta^{\varepsilon}_{\mathcal{C}})$
  - with $\varepsilon < 3/4$ for Neumann conditions,
  - with $\varepsilon < 1/4$ for Dirichlet conditions.
- Define a solution y of (1) as

$$y_t = S_t y_0 + \Delta_{\mathcal{C}} \int_0^t S_{t-r} N \Phi \ \mathrm{d}B^H_r .$$

Stochastic convolution for fBm

Theorem (Duncan, Pasik-Duncan, Maslowski '02 & '06)

If there is a constant $c > 0$ and $\delta \in (0, H)$ such that $\|G(r)\|_{HS} \leq cr^{-\delta}$ for all $r \in [0, t]$, then the stochastic integral $\int_0^t G(r)\ \mathrm{d}B_r^H$ is well-defined.

In our case:

$$\begin{aligned}\|\Delta_{\mathcal{C}} S_r N\Phi\|_{HS} &\leq \|\Phi\|_{HS} \|N\|_{L(L^2(\partial\mathcal{O});D(\Delta_{\mathcal{C}}^{\varepsilon}))} \|\Delta_{\mathcal{C}} S_r\|_{L(D(\Delta_{\mathcal{C}}^{\varepsilon});L^2(\mathcal{O}))} \\ &\leq \|\Phi\|_{HS} \|N\|_{L(L^2(\partial\mathcal{O});D(\Delta_{\mathcal{C}}^{\varepsilon}))}\, r^{\varepsilon-1}.\end{aligned}$$

Remark

In conclusion the convolution is well-defined if $1 - \varepsilon < H$, this means

- for any $H > 1/4$ in the Neumann case since $\varepsilon < 3/4$,
- for any $H > 3/4$ in the Dirichlet case since $\varepsilon < 1/4$.

Summary known results

$$\dot{y}_t = \Delta y$$
$$\mathcal{C} y_t = F(y_t)\, \dot{\mathbf{X}}_t.$$

- Da Prato and Zabczyk (additive noise):
  - $\mathbf{X}$ is a Brownian motion
  - Only Neumann conditions
- Duncan, Pasik-Duncan and Maslowski (additive noise):
  - $\mathbf{X}$ is a fractional Brownian motion with Hurst index $H > 1/4$
  - For $H > 3/4$ even Dirichlet conditions
- Schnaubelt and Veraar (multiplicative noise):
  - $\mathbf{X}$ is a Brownian motion
  - Equation in Banach spaces
- As far as we know, no work treated rough noise $\mathbf{X} = (X, \mathbb{X})$ on the boundary.

Motivation

- Transport models in chemical reactions [Wang, Zheng '05], [Brune, Duan, Schmalfuß '09].
- Primitive equations as a model for coupled atmosphere-ocean systems with wind driven boundary conditions

$$
\begin{aligned}
\mathrm{d}V + \nabla_{x,y} V + w(V) \cdot \partial_z V - \Delta V + \nabla_{x,y} P_s \, \mathrm{d}t &= H_f \, \mathrm{d}W \\
\operatorname{div}_{x,y} \bar{V} = 0, \; V(0) &= 0 \\
\partial_z V = 0 \text{ on } \Gamma_b, \partial_z V &= h_b \partial_t w \text{ on } \Gamma_u \\
V \text{ and } P_s \text{ periodic on } \Gamma_l
\end{aligned}
$$

$\rightsquigarrow$ Variant of the 3D Navier-Stokes, where the vertical component is averaged out [Lions, Temam, Wang '92 & '93], [Binz, Hieber, Hussein, Saal '22].

Controlled rough path

Abbreviation: $C(\mathcal{B}) = C([0,T];\mathcal{B})$.

Definition (γ-Hölder rough path)

A pair $\mathbf{X} = (X, \mathbb{X})$ is called a γ-Hölder rough path for $\gamma \in (1/3, 1/2]$ if $X \in C^\gamma(\mathbb{R}), \mathbb{X} \in C^{2\gamma}(\mathbb{R})$ and Chen's relation holds

$$\mathbb{X}_{t,s} - \mathbb{X}_{u,s} - \mathbb{X}_{t,u} = X_{u,s} \otimes X_{t,u}.$$

Definition (Gerasimovics, Hocquet, Nilsen '21)

We call a pair (y, y') a controlled rough path according to a monotone scale $(\mathcal{B}_\alpha)_{\alpha\in\mathbb{R}}$, if $(y, y') \in C(\mathcal{B}_\alpha) \times ((C(\mathcal{B}_{\alpha-\gamma}) \cap C^\gamma(\mathcal{B}_{\alpha-2\gamma}))$ and the remainder $R^y_{t,s} = y_{t,s} - y'_s X_{t,s}$ belongs to $C^\gamma(\mathcal{B}_{\alpha-\gamma}) \cap C^{2\gamma}(\mathcal{B}_{\alpha-2\gamma})$. The component y' is referred to as Gubinelli derivative of y and we write $(y, y') \in \mathcal{D}^{2\gamma}_{X,\alpha}$.

Rough convolution

Theorem (Gerasimovics, Hocquet, Nilssen '21)

Let $(y, y') \in \mathcal{D}^{2\gamma}_{X,\alpha}$. *Then the integral map*

$$(y, y') \mapsto (z, z') := \left(\int_0^{\cdot} S_{\cdot - r} y_r \, \mathrm{d}\mathbf{X}_r, y_\cdot \right)$$

maps $\mathcal{D}^{2\gamma}_{X,\alpha}$ *into* $\mathcal{D}^{2\gamma}_{X,\alpha+\theta}$ *for* $\theta < \gamma$.

Question: Can we extend this to treat boundary noise?
Goal: Make sense of the rough convolution

$$A \int_0^t S_{t-r} N F(y_r) \, \mathrm{d}\mathbf{X}_r.$$

Problems

Recall the convolution

$$A\int_0^t S_{t-r}NF(y_r)\ \mathrm{d}\mathbf{X}_r. \qquad (2)$$

- Which is the right scale to work in? $\rightsquigarrow$ Two scales are needed
  - For the boundary data $F(y_t)$.
  - For the solution y_t.
- What is the right Gubinelli derivative of the rough convolution (2)?
  - We expect $ANF(y)$.
  - But $NF(y)$ satisfies $\mathcal{C}NF(y)=F(y)$, so $NF(y)\notin D(A)$.

Solution: Work in Banach scales, which are constructed by extensions (and restrictions) of the operator A.

Fractional power spaces

$$\mathcal{B}_\alpha = \begin{cases} D(A^\alpha), & \alpha \geq 0 \\ \overline{L^2(\mathcal{O})}^{\|A^\alpha \cdot\|}, & \alpha < 0 \end{cases} \qquad \tilde{\mathcal{B}}_\alpha := H^{\alpha - 3/2}(\partial\mathcal{O})$$

with $\|\cdot\|_\alpha := \|A^\alpha \cdot\|$. For a second order differential operator with boundary operator $\mathcal{C}$, we get

$$\mathcal{B}_\alpha = \begin{cases} \{x \in H^{2\alpha}(\mathcal{O}) \mid \mathcal{C}u = 0\}, & 2\alpha > 3/2 \\ H^{2\alpha}(\mathcal{O}), & -1/2 < 2\alpha < 3/2 \\ \left(H^{-2\alpha}(\mathcal{O})\right)', & -3/2 < 2\alpha < -1/2 \\ \{x \in H^{-2\alpha}(\mathcal{O}) \mid \mathcal{C}u = 0\}', & 2\alpha < -3/2 \end{cases}.$$

Reminder: The Neumann map fulfills $N \in L(\tilde{\mathcal{B}}_\alpha, \mathcal{B}_\varepsilon)$ for $\alpha > 3/2$ and $\varepsilon < 3/4$.

Extrapolation operators

$$A_\alpha = \begin{cases} \mathcal{B}_\alpha\text{-Realization of } A, & \alpha \geq 0 \\ \text{Unique continuous extension of } A \text{ in } \mathcal{B}_{-1}, & \alpha = -1 \\ \mathcal{B}_\alpha\text{-Realization of } A_{-1}, & \alpha \in (-1, 0) \end{cases}$$

- Note that $\|Ax\|_{-1} = \|x\|$ for $x \in D(A)$ so A_{-1} is well-defined.
- The construction can be extended to define A_{-2}, A_{-3} and so on.
- Then for $\alpha > \beta$ we have $A_\alpha \in L(\mathcal{B}_{1+\alpha}, \mathcal{B}_\alpha)$ and $A_\alpha \subset A_\beta$.

Remark

This is a special case of the theory of Banach scales by [Amann '95].

Back to the problem

- For $\alpha > 3/2$ and $(y, y') \in \tilde{\mathcal{D}}^{2\gamma}_{X,\alpha}$ we have $(Ny, Ny') \in \mathcal{D}^{2\gamma}_{X,\varepsilon}$.
  Proof: Follows directly, since the Neumann-map satisfies $N \in \bigcap_{i=0}^{2} L(\tilde{\mathcal{B}}_{\alpha-i\gamma}, \mathcal{B}_{\varepsilon-i\gamma})$.
- For $\alpha > 3/2$ and $(y, y') \in \tilde{\mathcal{D}}^{2\gamma}_{X,\alpha}$ the integral $\mathcal{I}_t := \int_0^t S_{t-s} N y_s \, \mathrm{d}\mathbf{X}_s$ belongs to $D(A)$.
  Proof: Take $\theta := 1/3 + \delta < \gamma$ with $\delta > 0$ small and $\varepsilon := 3/4 - \delta$, so that we obtain $\varepsilon + \theta = 12/11 > 1$.

Conclusion

- So $\int_0^t S_{t-s} N y_s \, \mathrm{d}\mathbf{X}_s$ is well-defined in the sense of [Gerasimovics, Hocquet, Nilsen '21] with values in $D(A)$.
- Crucial is that ε can be chosen s.t. $1 - \varepsilon < \gamma$.
- It can be shown that $(A\mathcal{I}, A_{-\eta} N y) \in \mathcal{D}^{2\gamma}_{X,\alpha}$, for $\eta := 1 - \varepsilon$.

Extrapolation operators and CRP

- The structure of $(A\mathcal{I}, A_{-\eta}Ny)$ makes it difficult to get global existence using the methods in [Hesse, Neamțu '22].
- Consider instead $\int_0^t S_{t-s}A_{-\eta}Ny_s \; \mathrm{d}\mathbf{X}_s$.
- Set $-\sigma := -\eta - \gamma$, then $Ny'_t \in \mathcal{B}_{1-\sigma} = \mathcal{B}_{\varepsilon-\gamma}$ and $A_{-\sigma}Ny'_t$ is well-defined.

Lemma (Neamțu, Seitz '22)

For every $(y, y') \in \tilde{\mathcal{D}}^{2\gamma}_{X,\alpha}$ we have $(A_{-\sigma}Ny, A_{-\sigma}Ny') \in \mathcal{D}^{2\gamma}_{X,-\eta}$ with $\sigma := \eta + \gamma$.

Proof:

- $Ny_t \in \mathcal{B}_{1-\eta} \hookrightarrow \mathcal{B}_{1-\sigma}$ implies $A_{-\sigma}Ny_t = A_{-\eta}Ny_t \in \mathcal{B}_{-\eta}$.
- Similarly $Ny'_t \in \mathcal{B}_{\varepsilon-\gamma} = \mathcal{B}_{1-\sigma}$ implies $A_{-\sigma}Ny'_t \in \mathcal{B}_{-\sigma} = \mathcal{B}_{-\eta-\gamma}$.
- This leads to $(A_{-\sigma}Ny, A_{-\sigma}Ny') \in C(\mathcal{B}_{-\eta}) \times C(\mathcal{B}_{-\eta-\gamma})$.

Extrapolation operators and CRP

Theorem (Neamţu, Seitz '22)

For every $(y, y') \in \tilde{\mathcal{D}}^{2\gamma}_{X,\alpha}$ *we have* $(A_{-\sigma} Ny, A_{-\sigma} Ny') \in \mathcal{D}^{2\gamma}_{X,-\eta}$ *with* $\sigma := \eta + \gamma$.

Proof (continued): Now we need to control the Hölder-seminorms.

- For the Gubinelli derivative:

$$[A_{-\sigma} Ny']_{\gamma,-\eta-2\gamma} = [A_{-\eta-2\gamma} Ny']_{\gamma,-\eta-2\gamma} \lesssim [Ny']_{\gamma,1-\eta-2\gamma}$$
$$\Rightarrow A_{-\sigma} Ny' \in C^{\gamma}(\mathcal{B}_{-\eta-2\gamma})$$

- For the remainder $R^{Ny}_{t,s} \in \mathcal{B}_{1-\sigma}$ with $\theta \in \{\gamma, 2\gamma\}$:

$$\left[A_{-\sigma} R^{Ny}\right]_{\theta,-\eta-\theta} = \left[A_{-\sigma-(\theta-\gamma)} R^{Ny}\right]_{\theta,-\sigma-(\theta-\gamma)}$$
$$\lesssim \left[R^{Ny}\right]_{\theta,1-\sigma-(\theta-\gamma)} = \left[R^{Ny}\right]_{\theta,\varepsilon-\theta}$$
$$\Rightarrow A_{-\sigma} R^{Ny} \in C^{\gamma}(\mathcal{B}_{-\eta-\gamma}) \cap C^{2\gamma}(\mathcal{B}_{-\eta-2\gamma})$$

Summary

Lemma (Neamţu, Seitz '22)

For $t \in [0, T]$ and $(y, y') \in \tilde{\mathcal{D}}^{2\gamma}_{X,\alpha}$ we have

$$\left(\int_0^{\cdot} S_{\cdot - s} A_{-\sigma} N y_s \ \mathrm{d}\mathbf{X}_s, A_{-\sigma} N y \right) \in \mathcal{D}^{2\gamma}_{X,-\eta}.$$

- The original equation is now equivalent to one without boundary terms

$$\mathrm{d}y_t = A y_t \ \mathrm{d}t + A_{-\sigma} N F(y_t) \ \mathrm{d}\mathbf{X}_t \qquad \text{in } \mathcal{O}.$$

- For global existence $\rightsquigarrow$ investigate the nonlinearity $A_{-\sigma} N F(\cdot)$ to apply the theory in [Hesse, Neamţu '22].

Assumptions

(A1) There exists a $\delta > \eta + 3/2$ such that for any $\vartheta \in \{0, \gamma, 2\gamma\}$ the diffusion term $F : \mathcal{B}_{-\eta-\vartheta} \to \widetilde{\mathcal{B}}_{-\eta-\vartheta+\delta}$ is three times continuously differentiable with bounded derivatives.

(A2) Assume further the boundedness of the derivative of

$$DF(\cdot) \circ (A_{-\sigma} NF(\cdot)) : \mathcal{B}_{-\eta-\gamma} \to \widetilde{\mathcal{B}}_{-\eta-\gamma+\delta}.$$

$\rightsquigarrow$ F has to lift the spatial regularity, since we need that N maps to a strong solution of the problem

$$\mathcal{A} = 0, \ \mathcal{C}u = g.$$

Main Results

Conclusion. We obtain the semilinear PDE (without boundary noise)

$$\mathrm{d}y_t = Ay_t \ \mathrm{d}t + A_{-\sigma}NF(y_t) \ \mathrm{d}\mathbf{X}_t. \tag{3}$$

Theorem (Neamțu, Seitz '22)

- *Assume (A1). Then there exists for every initial condition* $y_0 \in \mathcal{B}_{-\eta}$ *a time* $T^* \leq T$ *and a unique solution* $(y, A_{-\sigma}NF(y)) \in \mathcal{D}^{2\gamma}_{X,-\eta}([0, T^*))$ *to* (3).
- *Assume (A1) and (A2). Then there exists for every initial condition* $y_0 \in \mathcal{B}_{-\eta}$ *a unique solution* $(y, A_{-\sigma}NF(y)) \in \mathcal{D}^{2\gamma}_{X,-\eta}([0, T])$ *to* (3).

The solution satisfies the mild formulation

$$y_t = S_t y_0 + \int_0^t S_{t-s}A_{-\sigma}NF(y_s) \ \mathrm{d}\mathbf{X}_s.$$

The Young case and Dirichlet boundary noise

The Young case $\tilde{\gamma} \in (3/4, 1)$:

- The solution map $\mathfrak{D}$ for the abstract problem with Dirichlet conditions $\mathcal{A}u = 0, u|_{\partial\mathcal{O}} = g$ maps $L^2(\partial\mathcal{O})$ into $D(A^\varepsilon)$ for $\varepsilon < 1/4$.
- Regarding the definition of the Young integral $F : \mathcal{B}_{-\eta-\vartheta} \to \widetilde{\mathcal{B}}_{-\eta-\vartheta+\delta}$ has to satisfy similar assumptions as above.

Theorem (Neamțu, Seitz '22)

There exists for every initial condition $y_0 \in \mathcal{B}_{-\eta}$ a unique mild solution $y \in C(\mathcal{B}_{-\eta}) \cap C^{\widetilde{\gamma}}(\mathcal{B}_{-\eta-\widetilde{\gamma}})$ that satisfies for all $t \in [0, T]$

$$y_t = S_t y_0 + \int_0^t S_{t-r} A_{-\sigma} \mathfrak{D} F(y_r) \, \mathrm{d}X_r,$$

where the integral is understood in the sense of Young.

Example of the diffusion coefficient

- We need an operator that lifts the spatial regularity $\rightsquigarrow$ Something like Δ^{ν}.
- Construct an operator $F : H^{-4}(\mathcal{O}) \to H^{\tilde{\delta}}(\partial\mathcal{O})$ for a small $\tilde{\delta} > 0$.

$$F : H^{-4}(\mathcal{O}) \to H^{\tilde{\delta}}(\partial\mathcal{O}), f \mapsto \gamma_{\partial} r_{\mathcal{O}} \Lambda^{\nu} e_{\mathcal{O}} f$$

- $\gamma_{\partial} : H^{\tilde{\delta}+1/2}(\mathcal{O}) \to H^{\tilde{\delta}}(\partial\mathcal{O})$ the trace operator.
- $\Lambda^{\nu} : H^{-4}(\mathbb{R}^d) \to H^{\tilde{\delta}+\frac{1}{2}}(\mathbb{R}^d), f \mapsto \mathcal{F}^{-1}(1+|\cdot|^2)^{\frac{\nu}{2}}\mathcal{F}f$ with $\nu := -9/2 - \tilde{\delta}$ to increase the spatial regularity on the full space.
- $r_{\mathcal{O}}$ retraction $e_{\mathcal{O}}$ coretraction to restrict $\mathbb{R}^d$ to $\mathcal{O}$ and extend this to $\mathbb{R}^d$, both linear and bounded.

Note $\mathcal{B}_{-\eta-2\gamma} \hookrightarrow \mathcal{B}_{-2} \hookrightarrow H^{-4}(\mathcal{O})$ and $H^{\tilde{\delta}}(\partial\mathcal{O}) = \tilde{\mathcal{B}}_{\tilde{\delta}+3/2}$, so
$F : \mathcal{B}_{-\eta-\vartheta} \to \widetilde{\mathcal{B}}_{\delta+3/2-\vartheta}$ for $\vartheta \in \{0, \gamma, 2\gamma\}$.
$\rightsquigarrow$ Same construction holds in the Dirichlet case, with $\nu := -11/2 - \tilde{\delta}$.

1 Introduction and Motivation

2 Preliminaries

3 Main Results

4 Outlook

Outlook

- Dynamical aspects, stabilization by boundary noise [Fellner, Sonner, Tang, Thuan '18]

$$\begin{aligned} \mathrm{d}u + (-\Delta u + u^3 - \beta u)\ \mathrm{d}t = 0 \qquad & \text{in } \mathcal{O} \times (0,T) \\ \mathrm{d}u + (\partial_\nu + \lambda u)\ \mathrm{d}t = \alpha u\ \mathrm{d}W_t \qquad & \text{on } \partial\mathcal{O} \times (0,T) \end{aligned}$$

- The primitive equation with rough noise on the boundary

$$\begin{aligned} \mathrm{d}V + \nabla_{x,y} V + w(V) \cdot \partial_z V - \Delta V + \nabla_{x,y} P_s \, \mathrm{d}t = H_f \, \mathrm{d}W \\ \operatorname{div}_{x,y} \bar{V} = 0, V(0) = 0 \\ \partial_z V = 0 \text{ on } \Gamma_b, \partial_z V = h_b \dot{\mathbf{X}} \text{ on } \Gamma_u \\ V \text{ and } P_s \text{ periodic on } \Gamma_l \end{aligned}$$

References I

T. E. Duncan, B. Pasik-Duncan, and B. Maslowski.
Fractional Brownian motion and stochastic equations in Hilbert spaces.
Stoch. Dyn., 2(2):225–250, 2002.

A. Gerasimovičs, A. Hocquet, and T. Nilssen.
Non-autonomous rough semilinear PDEs and the multiplicative sewing lemma.
J. Funct. Anal., 281(10):Paper No. 109200, 65, 2021.

R. Hesse and A. Neamţu.
Global solutions for semilinear rough partial differential equations.
Stoch. Dyn., 22(2):Paper No. 2240011, 18, 2022.

B. Maslowski.
Stability of semilinear equations with boundary and pointwise noise.
Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 22(1):55–93, 1995.

A. Neamţu and T. Seitz.
Stochastic evolution equations with rough boundary noise.
https://arxiv.org/pdf/2209.10348.pdf, 2022.

R. Schnaubelt and M. Veraar.
Stochastic equations with boundary noise.
In *Parabolic problems*, Progr. Nonlinear Differential Equations Appl. 80, pages 609–629. 2011.

Thank you for your attention!

More about global-in-time solutions

$$\mathrm{d}y_t = Ay_t \ \mathrm{d}t + G(y_t) \ \mathrm{d}\mathbf{X}_t$$

Problem: Even without boundary terms, bounds are harder to establish, since quadratic occur in estimating $(F(y), DF(y) \circ y')$.

- $\rightsquigarrow$ Use the structure of the expected solution $(y, G(y))$ to obtain better terms.
- Therefor stronger assumptions on G are needed, like $DG \circ G$ need a bounded derivative.
- Since here $G := A_{-\sigma}NF$, the assumption has to be adapted since $DF \circ F$ is not well-defined. That leads to the same condition for $DF \circ G$.
- For problems without boundary terms, see [Hesse, Neamțu '20 & '21] G is allowed to lose spatial regularity. $\rightsquigarrow$ Due to the Neumann map, now F has to lift regularity