Universität Konstanz

Stochastic evolution equations with rough boundary noise

Tim Seitz

joint work with Alexandra Neamțu

SPDEvent Bielefeld University, 18. July 2023

General Setting

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with smooth boundary, and consider

$$\begin{aligned} \dot{y}_t &= \mathcal{A}y_t + f(y_t) \quad \text{in } \mathcal{O}, \\ \mathcal{C}y_t &= F(y_t)\,\dot{\mathbf{X}}_t \qquad \text{on } \partial\mathcal{O}. \end{aligned}$$

$\mathcal{A}$ is a second order operator, $\mathcal{C}$ is a Neumann type boundary condition and $A : D(A) \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$ its L^2-realization.

General Setting

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with smooth boundary, and consider

$$\begin{aligned} \dot{y}_t &= \mathcal{A} y_t + f(y_t) \quad \text{in } \mathcal{O}, \\ \mathcal{C} y_t &= F(y_t)\, \dot{\mathbf{X}}_t \qquad \text{on } \partial\mathcal{O}. \end{aligned}$$

$\mathcal{A}$ is a second order operator, $\mathcal{C}$ is a Neumann type boundary condition and $A : D(A) \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$ its L^2-realization.
We expect a solution of the form

$$y_t = S_t y_0 + \int_0^t S_{t-r} f(y_r)\, \mathrm{d}r + A \int_0^t S_{t-r} N F(y_r)\, \mathrm{d}\mathbf{X}_r.$$

Where $N : L^2(\partial\mathcal{O}) \to H^{3/2}(\mathcal{O})$ is the solution operator to the elliptic boundary value problem $\mathcal{A}u = 0,\ \mathcal{C}u = g$.

General Setting

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with smooth boundary, and consider

$$\begin{aligned} \dot{y}_t &= \mathcal{A} y_t + f(y_t) \quad &\text{in } \mathcal{O}, \\ \mathcal{C} y_t &= F(y_t)\, \dot{\mathbf{X}}_t \quad &\text{on } \partial\mathcal{O}. \end{aligned}$$

$\mathcal{A}$ is a second order operator, $\mathcal{C}$ is a Neumann type boundary condition and $A : D(A) \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$ its L^2-realization.
We expect a solution of the form

$$y_t = S_t y_0 + \int_0^t S_{t-r} f(y_r)\, \mathrm{d}r + A \int_0^t S_{t-r} N F(y_r)\, \mathrm{d}\mathbf{X}_r.$$

Where $N : L^2(\partial\mathcal{O}) \to H^{3/2}(\mathcal{O})$ is the solution operator to the elliptic boundary value problem $\mathcal{A}u = 0,\ \mathcal{C}u = g$.

$\rightsquigarrow$ We need to make sense of the rough convolution

Summary known results

- Da Prato and Zabczyk ’93:
  - **X** is a Brownian motion, additive noise
  - Only Neumann conditions
- Duncan, Pasik-Duncan and Maslowski ’02 & ’06:
  - **X** is a fractional Brownian motion, additive noise
  - Neumann with Hurst index $H > 1/4$, for $H > 3/4$ even Dirichlet conditions
- Schnaubelt and Veraar ’11:
  - **X** is a Brownian motion, multiplicative noise
  - Only Neumann Conditions, non-autonomous equation in Banach spaces,
- As far as we know, no work treated rough noise $\mathbf{X} = (X, \mathbb{X})$ on the boundary.

Controlled rough path approach

Abbreviation: $C(\mathcal{B}) = C([0,T];\mathcal{B})$ where $\mathcal{B}$ is a Banach space

- $\mathbf{X} = (X, \mathbb{X})$ is a γ-Hölder rough path with $\gamma \in (1/3, 1/2]$
- $(\mathcal{B}_\alpha)_{\alpha\in\mathbb{R}}$ monotone scale of function spaces for example $(H^\alpha(\mathcal{O}))_{\alpha\in\mathbb{R}}$
- $(y, y') \in C(\mathcal{B}_\alpha) \times ((C(\mathcal{B}_{\alpha-\gamma}) \cap C^\gamma(\mathcal{B}_{\alpha-2\gamma}))$ such that $R^y_{t,s} = y_{t,s} - y'_s X_{t,s}$ belongs to $C^\gamma(\mathcal{B}_{\alpha-\gamma}) \cap C^{2\gamma}(\mathcal{B}_{\alpha-2\gamma})$
  $\rightsquigarrow$ controlled rough path in the sense of [Gerasimovics, Hocquet, Nilssen '21]. Notation: $(y, y') \in \mathcal{D}^{2\gamma}_{X,\alpha}$

Theorem (Gerasimovics, Hocquet, Nilssen '21)

Let $(y, y') \in \mathcal{D}^{2\gamma}_{X,\alpha}$. Then the integral map

$$(y, y') \mapsto (z, z') := \left(\int_0^\cdot S_{\cdot - r} y_r \, \mathrm{d}\mathbf{X}_r, y_\cdot \right)$$

maps $\mathcal{D}^{2\gamma}_{X,\alpha}$ into $\mathcal{D}^{2\gamma}_{X,\alpha+\theta}$ for $\theta < \gamma$.

Difficulties

Goal: Make sense of the rough convolution

$$A \int_0^t S_{t-r} N F(y_r)\ \mathrm{d}\mathbf{X}_r \qquad (1)$$

in the controlled rough path setting.

Difficulties

Goal: Make sense of the rough convolution

$$A \int_0^t S_{t-r} N F(y_r) \, \mathrm{d}\mathbf{X}_r \qquad (1)$$

in the controlled rough path setting.

Problem: Expected Gubinelli derivative $ANF(y)$ of (1) is not well-defined, since $\mathcal{C}NF(y) = F(y)$, so $NF(y) \notin D(A)$.

Difficulties

Goal: Make sense of the rough convolution

$$A \int_0^t S_{t-r} N F(y_r) \, \mathrm{d}\mathbf{X}_r \tag{1}$$

in the controlled rough path setting.

Problem: Expected Gubinelli derivative $ANF(y)$ of (1) is not well-defined, since $\mathcal{C}NF(y) = F(y)$, so $NF(y) \notin D(A)$.

Solution: Work with extensions (and restrictions) of the operator A.

Fractional power spaces and extrapolation operators

We introduce the function spaces

$$\mathcal{B}_\alpha = \begin{cases} D(A^\alpha), & \alpha \geq 0 \\ \overline{L^2(\mathcal{O})}^{\|A^\alpha \cdot\|}, & \alpha < 0 \end{cases} \qquad \tilde{\mathcal{B}}_\alpha := H^{\alpha - 3/2}(\partial\mathcal{O})$$

with $\|\cdot\|_\alpha := \|A^\alpha \cdot\|$.

Note: $N \in L(\widetilde{\mathcal{B}}_\alpha, \mathcal{B}_\varepsilon)$ if $\alpha > 3/2$, for any $2\varepsilon < 3/2$.

Fractional power spaces and extrapolation operators

We introduce the function spaces

$$\mathcal{B}_\alpha = \begin{cases} D(A^\alpha), & \alpha \geq 0 \\ \overline{L^2(\mathcal{O})}^{\|A^\alpha \cdot\|}, & \alpha < 0 \end{cases} \qquad \tilde{\mathcal{B}}_\alpha := H^{\alpha - 3/2}(\partial\mathcal{O})$$

with $\|\cdot\|_\alpha := \|A^\alpha \cdot\|$.

Note: $N \in L(\widetilde{\mathcal{B}}_\alpha, \mathcal{B}_\varepsilon)$ if $\alpha > 3/2$, for any $2\varepsilon < 3/2$.

$$A_\alpha = \begin{cases} \mathcal{B}_\alpha\text{-realization of } A, & \alpha \geq 0 \\ \text{unique continuous extension of } A \text{ in } \mathcal{B}_{-1}, & \alpha = -1 \\ \mathcal{B}_\alpha\text{-realization of } A_{-1}, & \alpha \in (-1, 0) \end{cases}$$

Properties (Amann '95)

- $A_\alpha \in L(\mathcal{B}_{1+\alpha}, \mathcal{B}_\alpha)$ and $A_\alpha \subset A_\beta$ for $\alpha > \beta$
- A_{-1} is called the extrapolated operator of A.
- Since $Ny_t \notin D(A)$ we need those extensions to define $A_{-\eta} N y_t$ for $\eta := 1 - \varepsilon$.

Extrapolation operators and CRP

Lemma (Neamțu, S. ’22)

For $t \in [0,T]$ and $(y, y') \in \tilde{\mathcal{D}}^{2\gamma}_{X,\alpha}$ we have

$$\left(A \int_0^{\cdot} S_{\cdot - s} N y_s \;\mathrm{d}\mathbf{X}_s, A_{-\eta} N y \right) \in \mathcal{D}^{2\gamma}_{X,\alpha},$$

with $\eta := 1 - \varepsilon$ if $\varepsilon > 1 - \gamma$. Furthermore

$$\left(\int_0^{\cdot} S_{\cdot - s} A_{-\sigma} N y_s \;\mathrm{d}\mathbf{X}_s, A_{-\sigma} N y \right) \in \mathcal{D}^{2\gamma}_{X,\alpha},$$

for $\sigma := \eta + \gamma$.

Extrapolation operators and CRP

Lemma (Neamțu, S. '22)

For $t \in [0,T]$ and $(y, y') \in \tilde{\mathcal{D}}^{2\gamma}_{X,\alpha}$ we have

$$\left(A \int_0^{\cdot} S_{\cdot - s} N y_s \ \mathrm{d}\mathbf{X}_s, A_{-\eta} N y \right) \in \mathcal{D}^{2\gamma}_{X,\alpha},$$

with $\eta := 1 - \varepsilon$ if $\varepsilon > 1 - \gamma$. Furthermore

$$\left(\int_0^{\cdot} S_{\cdot - s} A_{-\sigma} N y_s \ \mathrm{d}\mathbf{X}_s, A_{-\sigma} N y \right) \in \mathcal{D}^{2\gamma}_{X,\alpha},$$

for $\sigma := \eta + \gamma$.

The original equation is now equivalent to one without boundary terms

$$\mathrm{d}y_t = (A y_t + f(y_t)) \ \mathrm{d}t + A_{-\sigma} N F(y_t) \ \mathrm{d}\mathbf{X}_t, \qquad y(0) = y_0$$

Main Results

(A1) There exists $\delta > \eta + 3/2$ such that for any $\vartheta \in \{0, \gamma, 2\gamma\}$ the diffusion term $F : \mathcal{B}_{-\eta-\vartheta} \to \widetilde{\mathcal{B}}_{-\eta-\vartheta+\delta}$ is three times continuously differentiable with bounded derivatives.

(A2) Assume further the boundedness of the derivative of

$$DF(\cdot) \circ (A_{-\sigma} NF(\cdot)) : \mathcal{B}_{-\eta-\gamma} \to \widetilde{\mathcal{B}}_{-\eta-\gamma+\delta}.$$

Theorem (Neamțu, S. '22)

- Assume (A1). Then there exists for every $y_0 \in \mathcal{B}_{-\eta}$ a time $T^* \leq T$ and a unique solution $(y, A_{-\sigma} NF(y)) \in \mathcal{D}^{2\gamma}_{X,-\eta}([0, T^*))$.
- Assume (A1) and (A2). Then there exists for every initial condition $y_0 \in \mathcal{B}_{-\eta}$ a unique solution $(y, A_{-\sigma} NF(y)) \in \mathcal{D}^{2\gamma}_{X,-\eta}([0, T])$.

Some remarks

- If $\gamma > {}^{1}\!/_{2}$, no rough path theory is needed since the integral is well-defined in the sense of Young.
  $\rightsquigarrow$ Then even Dirichlet conditions are possible to treat, provided that $\gamma > {}^{3}\!/_{4}$.
- An example for a possible nonlinearity is a modified version of $\Delta^{-\nu}$ for some $\nu > 0$.
- Due to pathwise solutions which are global-in-time we have also the existence of a random dynamical system.

Outlook

- Investigate more dynamical aspects (Existence of random attractors, stability etc.)

Outlook

- Investigate more dynamical aspects (Existence of random attractors, stability etc.)
- Primitive equations as a model for coupled atmosphere-ocean systems with wind driven boundary conditions [Binz, Hieber, Hussein, Saal '22]

$$\mathrm{d}V + \nabla_{x,y}V + w(V)\cdot\partial_z V - \Delta V + \nabla_{x,y}P_s\,\mathrm{d}t = H_f\,\mathrm{d}W$$
$$\operatorname{div}_{x,y}\bar{V} = 0,\ V(0) = 0$$
$$\partial_z V = 0 \text{ on } \Gamma_b,\ \partial_z V = h_b\partial_t w \text{ on } \Gamma_u$$
$$V \text{ and } P_s \text{ periodic on } \Gamma_l$$

Outlook

- Investigate more dynamical aspects (Existence of random attractors, stability etc.)
- Primitive equations as a model for coupled atmosphere-ocean systems with wind driven boundary conditions [Binz, Hieber, Hussein, Saal '22]

$$
\begin{aligned}
\mathrm{d}V + \nabla_{x,y} V + w(V) \cdot \partial_z V - \Delta V + \nabla_{x,y} P_s \, \mathrm{d}t &= H_f \, \mathrm{d}W \\
\operatorname{div}_{x,y} \bar{V} &= 0, V(0) = 0 \\
\partial_z V = 0 \text{ on } \Gamma_b, \partial_z V &= h_b \partial_t w \text{ on } \Gamma_u \\
V \text{ and } P_s &\text{ periodic on } \Gamma_l
\end{aligned}
$$

- 2D Navier-Stokes with boundary noise [Agresti, Luongo '23]

References

T. E. Duncan, B. Pasik-Duncan, and B. Maslowski.
Fractional Brownian motion and stochastic equations in Hilbert spaces.
Stoch. Dyn., 2(2):225–250, 2002.

A. Gerasimovičs, A. Hocquet, and T. Nilssen.
Non-autonomous rough semilinear PDEs and the multiplicative sewing lemma.
J. Funct. Anal., 281(10):Paper No. 109200, 65, 2021.

R. Hesse and A. Neamțu.
Global solutions for semilinear rough partial differential equations.
Stoch. Dyn., 22(2):Paper No. 2240011, 18, 2022.

B. Maslowski.
Stability of semilinear equations with boundary and pointwise noise.
Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 22(1):55–93, 1995.

A. Neamțu and T. Seitz.
Stochastic evolution equations with rough boundary noise.
https://arxiv.org/pdf/2209.10348.pdf, 2022.

R. Schnaubelt and M. Veraar.
Stochastic equations with boundary noise.
In Parabolic problems, Progr. Nonlinear Differential Equations Appl. 80, pages 609–629. 2011.

Thank you for your attention!

More about global-in-time solutions

$$\mathrm{d}y_t = Ay_t \ \mathrm{d}t + G(y_t) \ \mathrm{d}\mathbf{X}_t$$

Problem: Even without boundary terms, bounds are harder to establish, since quadratic occur in estimating $(F(y), DF(y) \circ y')$.

- $\rightsquigarrow$ Use the structure of the expected solution $(y, G(y))$ to obtain better terms.
- Therefor stronger assumptions on G are needed, like $DG \circ G$ need a bounded derivative.
- Since here $G := A_{-\sigma}NF$, the assumption has to be adapted since $DG \circ G$ is not well-defined. That leads to the same condition for $DF \circ G$.
- For problems without boundary terms, see [Hesse, Neamțu ’20 & ’21], G is allowed to lose spatial regularity.
  $\rightsquigarrow$ Due to the Neumann map, now F has to lift regularity

Example of the diffusion coefficient

- We need an operator that lifts the spatial regularity $\rightsquigarrow$ Something like Δ^ν.
- Construct an operator $F : H^{-4}(\mathcal{O}) \to H^{\tilde{\delta}}(\partial\mathcal{O})$ for a small $\tilde{\delta} > 0$.

$$F : H^{-4}(\mathcal{O}) \to H^{\tilde{\delta}}(\partial\mathcal{O}), f \mapsto \gamma_\partial r_\mathcal{O} \Lambda^\nu e_\mathcal{O} f$$

- $\gamma_\partial : H^{\tilde{\delta}+1/2}(\mathcal{O}) \to H^{\tilde{\delta}}(\partial\mathcal{O})$ the trace operator.
- $\Lambda^\nu : H^{-4}(\mathbb{R}^d) \to H^{\tilde{\delta}+\frac{1}{2}}(\mathbb{R}^d), f \mapsto \mathcal{F}^{-1}(1+|\cdot|^2)^{\frac{\nu}{2}}\mathcal{F}f$ with $\nu := -9/2 - \tilde{\delta}$ to increase the spatial regularity on the full space.
- $r_\mathcal{O}$ retraction $e_\mathcal{O}$ coretraction to restrict $\mathbb{R}^d$ to $\mathcal{O}$ and extend this to $\mathbb{R}^d$, both linear and bounded.

Note $\mathcal{B}_{-\eta-2\gamma} \hookrightarrow \mathcal{B}_{-2} \hookrightarrow H^{-4}(\mathcal{O})$ and $H^{\tilde{\delta}}(\partial\mathcal{O}) = \tilde{\mathcal{B}}_{\tilde{\delta}+3/2}$, so

$F : \mathcal{B}_{-\eta-\vartheta} \to \widetilde{\mathcal{B}}_{\delta+3/2-\vartheta}$ for $\vartheta \in \{0, \gamma, 2\gamma\}$.

$\rightsquigarrow$ Same construction holds in the Dirichlet case, with $\nu := -11/2 - \tilde{\delta}$.