Stochastic evolution equations with rough boundary noise

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joint work with Alexandra Neamțu

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General Setting

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with smooth boundary, and consider

$$\begin{aligned} \dot{y}_t &= \mathcal{A} y_t + f(y_t) \quad \text{in } \mathcal{O}, \\ \mathcal{C} y_t &= F(y_t)\, \dot{\mathbf{X}}_t \qquad \text{on } \partial\mathcal{O}. \end{aligned}$$

$\mathcal{A}$ is a second order operator, $\mathcal{C}$ is a Neumann type boundary condition and $A : D(A) \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$ its L^2-realization.

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We expect a solution of the form

$$y_t = S_t y_0 + \int_0^t S_{t-r} f(y_r)\ \mathrm{d}r + A \int_0^t S_{t-r} N F(y_r)\ \mathrm{d}\mathbf{X}_r.$$

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$\rightsquigarrow$ We need to make sense of the rough convolution

Summary known results

- Da Prato and Zabczyk '93:
  - $\mathbf{X}$ is a Brownian motion, additive noise
  - Only Neumann conditions
- Duncan, Pasik-Duncan and Maslowski '02 & '06:
  - $\mathbf{X}$ is a fractional Brownian motion, additive noise
  - Neumann with Hurst index $H > 1/4$, for $H > 3/4$ even Dirichlet conditions
- Schnaubelt and Veraar '11:
  - $\mathbf{X}$ is a Brownian motion, multiplicative noise
  - Only Neumann Conditions, non-autonomous equation in Banach spaces,
- As far as we know, no work treated rough noise $\mathbf{X} = (X, \mathbb{X})$ on the boundary.

Controlled rough path approach

Abbreviation: $C(\mathcal{B}) = C([0,T];\mathcal{B})$ where $\mathcal{B}$ is a Banach space

- $\mathbf{X} = (X, \mathbb{X})$ is a γ-Hölder rough path with $\gamma \in (1/3, 1/2]$
- $(\mathcal{B}_\alpha)_{\alpha\in\mathbb{R}}$ monotone scale of function spaces for example $(H^\alpha(\mathcal{O}))_{\alpha\in\mathbb{R}}$
- $(y, y') \in C(\mathcal{B}_\alpha) \times ((C(\mathcal{B}_{\alpha-\gamma}) \cap C^\gamma(\mathcal{B}_{\alpha-2\gamma}))$ such that $R^y_{t,s} = y_{t,s} - y'_s X_{t,s}$ belongs to $C^\gamma(\mathcal{B}_{\alpha-\gamma}) \cap C^{2\gamma}(\mathcal{B}_{\alpha-2\gamma})$ $\rightsquigarrow$ controlled rough path in the sense of [Gerasimovics, Hocquet, Nilssen '21]. Notation: $(y, y') \in \mathcal{D}^{2\gamma}_{X,\alpha}$

Theorem (Gerasimovics, Hocquet, Nilssen '21)

Let $(y, y') \in \mathcal{D}^{2\gamma}_{X,\alpha}$. Then the integral map

$$(y, y') \mapsto (z, z') := \left(\int_0^\cdot S_{\cdot-r} y_r \, \mathrm{d}\mathbf{X}_r, y_\cdot\right)$$

maps $\mathcal{D}^{2\gamma}_{X,\alpha}$ into $\mathcal{D}^{2\gamma}_{X,\alpha+\theta}$ for $\theta < \gamma$.

Difficulties

Goal: Make sense of the rough convolution

$$A \int_0^t S_{t-r} NF(y_r) \, \mathrm{d}\mathbf{X}_r. \tag{1}$$

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Solution: Work with extensions (and restrictions) of the operator A.

Fractional power spaces and extrapolation operators

We introduce the function spaces

$$\mathcal{B}_\alpha = \begin{cases} D(A^\alpha), & \alpha \geq 0 \\ \overline{L^2(\mathcal{O})}^{\|A^\alpha \cdot\|}, & \alpha < 0 \end{cases} \qquad \tilde{\mathcal{B}}_\alpha := H^{\alpha - 3/2}(\partial\mathcal{O})$$

with $\|\cdot\|_\alpha := \|A^\alpha \cdot\|$.
Note: $N \in L(\widetilde{\mathcal{B}}_\alpha, \mathcal{B}_\varepsilon)$ if $\alpha > 3/2$, for any $2\varepsilon < 3/2$.

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$$A_\alpha = \begin{cases} \mathcal{B}_\alpha\text{-realization of } A, & \alpha \geq 0 \\ \text{unique continuous extension of } A \text{ in } \mathcal{B}_{-1}, & \alpha = -1 \\ \mathcal{B}_\alpha\text{-realization of } A_{-1}, & \alpha \in (-1, 0) \end{cases}$$

Properties (Amann '95)

- $A_\alpha \in L(\mathcal{B}_{1+\alpha}, \mathcal{B}_\alpha)$ and $A_\alpha \subset A_\beta$ for $\alpha > \beta$
- A_{-1} is called the extrapolated operator of A.
- Since $Ny_t \notin D(A)$ we need those extensions to define $A_{-\eta} N y_t$ for $\eta := 1 - \varepsilon$.

Lemma (Neamțu, S. '22)

For $t \in [0, T]$ and $(y, y') \in \tilde{\mathcal{D}}^{2\gamma}_{X,\alpha}$ we have

$$\left(A \int_0^{\cdot} S_{\cdot - s} N y_s \; \mathrm{d}\mathbf{X}_s, A_{-\eta} N y \right) \in \mathcal{D}^{2\gamma}_{X,\alpha},$$

with $\eta := 1 - \varepsilon$ if $\varepsilon > 1 - \gamma$. Furthermore

$$\left(\int_0^{\cdot} S_{\cdot - s} A_{-\sigma} N y_s \; \mathrm{d}\mathbf{X}_s, A_{-\sigma} N y \right) \in \mathcal{D}^{2\gamma}_{X,\alpha},$$

for $\sigma := \eta + \gamma$.

Extrapolation operators and CRP

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for $\sigma := \eta + \gamma$.

The original equation is now equivalent to one without boundary terms

$$\mathrm{d}y_t = (A y_t + f(y_t)) \, \mathrm{d}t + A_{-\sigma} N F(y_t) \, \mathrm{d}\mathbf{X}_t, \qquad y(0) = y_0$$

Main Results

(A1) There exists $\delta > \eta + 3/2$ such that for any $\vartheta \in \{0, \gamma, 2\gamma\}$ the diffusion term $F : \mathcal{B}_{-\eta-\vartheta} \to \widetilde{\mathcal{B}}_{-\eta-\vartheta+\delta}$ is three times continuously differentiable with bounded derivatives.

(A2) Assume further the boundedness of the derivative of

$$DF(\cdot) \circ (A_{-\sigma} NF(\cdot)) : \mathcal{B}_{-\eta-\gamma} \to \widetilde{\mathcal{B}}_{-\eta-\gamma+\delta}.$$

Theorem (Neamțu, S. '22)

- *Assume (A1). Then there exists for every* $y_0 \in \mathcal{B}_{-\eta}$ *a time* $T^* \leq T$ *and a unique solution* $(y, A_{-\sigma}NF(y)) \in \mathcal{D}^{2\gamma}_{X,-\eta}([0, T^*])$.
- *Assume (A1) and (A2). Then there exists for every initial condition* $y_0 \in \mathcal{B}_{-\eta}$ *a unique solution* $(y, A_{-\sigma}NF(y)) \in \mathcal{D}^{2\gamma}_{X,-\eta}([0, T])$.

Closing remarks

- If $\gamma > 1/2$, no rough path theory is needed since the integral is well-defined in the sense of Young.
  $\rightsquigarrow$ Then even Dirichlet conditions are possible to treat, provided that $\gamma > 3/4$.
- An example for a possible nonlinearity is a modified version of $\Delta^{-\nu}$ for some $\nu > 0$.
- Due to pathwise solutions which are global-in-time we have also the existence of a random dynamical system.

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Thank you for your attention!

More about global-in-time solutions

$$\mathrm{d}y_t = Ay_t \ \mathrm{d}t + G(y_t) \ \mathrm{d}\mathbf{X}_t$$

Problem: Even without boundary terms, bounds are harder to establish, since quadratic occur in estimating $(F(y), DF(y) \circ y')$.

- $\rightsquigarrow$ Use the structure of the expected solution $(y, G(y))$ to obtain better terms.
- Therefor stronger assumptions on G are needed, like $DG \circ G$ need a bounded derivative.
- Since here $G := A_{-\sigma}NF$, the assumption has to be adapted since $DG \circ G$ is not well-defined. That leads to the same condition for $DF \circ G$.
- For problems without boundary terms, see [Hesse, Neamţu '20 & '21], G is allowed to lose spatial regularity.
  $\rightsquigarrow$ Due to the Neumann map, now F has to lift regularity

Example of the diffusion coefficient

- We need an operator that lifts the spatial regularity $\rightsquigarrow$ Something like Δ^{ν}.
- Construct an operator $F : H^{-4}(\mathcal{O}) \to H^{\tilde{\delta}}(\partial\mathcal{O})$ for a small $\tilde{\delta} > 0$.

$$F : H^{-4}(\mathcal{O}) \to H^{\tilde{\delta}}(\partial\mathcal{O}), f \mapsto \gamma_{\partial} r_{\mathcal{O}} \Lambda^{\nu} e_{\mathcal{O}} f$$

- $\gamma_{\partial} : H^{\tilde{\delta}+1/2}(\mathcal{O}) \to H^{\tilde{\delta}}(\partial\mathcal{O})$ the trace operator.
- $\Lambda^{\nu} : H^{-4}(\mathbb{R}^d) \to H^{\tilde{\delta}+\frac{1}{2}}(\mathbb{R}^d), f \mapsto \mathcal{F}^{-1}(1+|\cdot|^2)^{\frac{\nu}{2}}\mathcal{F}f$ with $\nu := -9/2 - \tilde{\delta}$ to increase the spatial regularity on the full space.
- $r_{\mathcal{O}}$ retraction $e_{\mathcal{O}}$ coretraction to restrict $\mathbb{R}^d$ to $\mathcal{O}$ and extend this to $\mathbb{R}^d$, both linear and bounded.

Note $\mathcal{B}_{-\eta-2\gamma} \hookrightarrow \mathcal{B}_{-2} \hookrightarrow H^{-4}(\mathcal{O})$ and $H^{\tilde{\delta}}(\partial\mathcal{O}) = \tilde{\mathcal{B}}_{\tilde{\delta}+3/2}$, so
$F : \mathcal{B}_{-\eta-\vartheta} \to \widetilde{\mathcal{B}}_{\delta+3/2-\vartheta}$ for $\vartheta \in \{0, \gamma, 2\gamma\}$.
$\rightsquigarrow$ Same construction holds in the Dirichlet case, with $\nu := -11/2 - \tilde{\delta}$.

Motivation for noise on the boundary

- Transport models in chemical reactions [Wang, Zheng '05], [Brune, Duan, Schmalfuß '09].
- Primitive equations as a model for coupled atmosphere-ocean systems with wind driven boundary conditions

$$
\begin{aligned}
\mathrm{d}V + \nabla_{x,y} V + w(V) \cdot \partial_z V - \Delta V + \nabla_{x,y} P_s \,\mathrm{d}t &= H_f \,\mathrm{d}W \\
\operatorname{div}_{x,y} \bar{V} &= 0, \; V(0) = 0 \\
\partial_z V = 0 \text{ on } \Gamma_b, \partial_z V &= h_b \partial_t w \text{ on } \Gamma_u \\
V \text{ and } P_s &\text{ periodic on } \Gamma_l
\end{aligned}
$$

⇝ Variant of the 3D Navier-Stokes, where the vertical component is averaged out [Lions, Temam, Wang '92 & '93], [Binz, Hieber, Hussein, Saal '22].