REAL ALGEBRAIC GEOMETRY LECTURE NOTES (17: 15/06/15 - CORRECTED ON 17/06/2019)

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Contents

1. Kaplansky's Embedding Theorem

In the last lecture we showed that

(i) the value group of a real closed field K is isomorphic (as an ordered group) to a subgroup of $(K^{>0}, \cdot, 1, <)$.

(ii) if K is a real closed field, then every maximal Archimedean subfield of K is isomorphic to $\overline{K}$ (with respect to the natural valuation), and there exist such Archimedean subfields (lemma of Zorn). Therefore the residue field $\overline{K}$ is isomorphic to some subfield of K.

(iii) If $k[G]$ is a group ring, then $\text{ff}(k[G]) = k(G) = k(t^g : g \in G)$ is the smallest subfield of $k((G))$ generated by $k \cup \{t^g : g \in G\}$.

Theorem 1.1. *(Kaplansky's "sandwiching" or embedding theorem for rcf) Let K be a real closed field, G its value group and k its residue field. Then there exists a subfield of K isomorphic to $k(G)^{rc}$.*
Moreover, every such isomorphism extends to an embedding of K into $k((G))$,

$$\begin{array}{ccc} K & \xhookrightarrow{\ \mu\ } & k((G)) \\ \Big| & & \Big| \\ l(\mathbb{B})^{\mathrm{rc}} & \xrightarrow[\sim]{\ \mu_0\ } & k(G)^{\mathrm{rc}} \end{array}$$

i.e. K is isomorphic to a subfield $\mu(K)$ such that $k(G)^{rc} \subseteq \mu(K) \subseteq k((G))$.

Proof. Let $l \subseteq K$ be a subfield isomorphic to k and let $\mathbb{B}$ be a subgroup isomorphic to G. More precisely, $\mathbb{B}$ is a multiplicative subgroup of $(K^{>0}, \cdot, 1, <)$ isomorphic to the multiplicative subgroup $\{t^g : g \in G\}$ of monomials in $k((G))$. Consider the subfield of K generated by $l \cup \mathbb{B}$, i.e. the subfield $l(\mathbb{B})$ and we take its relative algebraic closure in K.
It is clear that $\exists$ isomorphism $\mu_0 : l(\mathbb{B})^{\mathrm{rc}} \to k(G)^{\mathrm{rc}}$.

Claim 1: the extension $l(\mathbb{B})^{\mathrm{rc}} \subseteq K$ is immediate.
This is because the residue field of a real closure equals the real closure of the residue field equals the residue field of K. Also the value group of the real closure is the divisible hull of the value group $= G$. So the extension is value group preserving and residue field preserving. Therefore the extension is immediate.
Now consider the collection of all pairs (M, μ) where M is a real closed subfield of K containing $l(\mathbb{B})^{\mathrm{rc}}$ and $\mu : M \hookrightarrow k((G))$ is an embedding of M extending μ_0. We partially order this collection the obvious way, i.e.

$$(M_1, \mu_1) \leqslant (M_2, \mu_2) :\Leftrightarrow M_1 \subseteq M_2, \mu_{2|M_1} = \mu_1.$$

It is clear that every chain $\mathcal{C}$ in this collection has an upper bound in it, namely $\bigcup \mathcal{C}$. So the hypothesis of Zorn's lemma is verified. Therefore, we find some maximal element (M, μ).

$$\begin{array}{ccc} K & \xrightarrow{\ \mu\ } & k((G)) \\ \text{immediate}\Big| & & \Big| \\ l(\mathbb{B})^{\mathrm{rc}} & \underset{\sim}{\xrightarrow{\ \mu_0\ }} & k(G)^{\mathrm{rc}} \end{array}$$

Claim 2: $M = K$.
We argue by contradiction. If this is not the case, let $y \in K \backslash M$. Note that y is transcendental over M. Also since $K \supseteq M$ is immediate, y is a pseudo-limit of a pseudo-Cauchy sequence $\{y_\alpha\}_{\alpha \in S} \subset M$ without a limit in M. Set $z_\alpha := \mu(y_\alpha)$, so $\{z_\alpha\}_{\alpha \in S} \subset k((G))$ is a pseudo-Cauchy sequence and $k((G))$ is pseudo-complete, so choose $z \in k((G))$ a pseudo-limit of $\{z_\alpha\}_{\alpha \in S}$.

Claim 3: z is transcendental over $\mu(M)$.
This is because $z \notin \mu(M)$. Otherwise $\mu^{-1}(z) \in M$ would be a pseudo-limit of $\{y_\alpha\}_{\alpha \in S} = \{\mu^{-1}(z_\alpha)\}_{\alpha \in S}$ in M, a contradiction.
Therefore $M(y) \cong \mu(M)(z)$ as fields and $M(y)^{\mathrm{rc}} \cong \mu(M)(z)^{\mathrm{rc}}$, contradicting the maximality of (M, μ).

□

Chapter III: Convex valuations on ordered fields:

2. Convex valuations

Let K be a non-Archimedean ordered field. Let v be its non-trivial natural valuation with valuation ring K_v and valuation ideal I_v.

Definition 2.1. Let w be a valuation on K. We say that w **is compatible with the order** (or **convex**) if $\forall\, a, b \in K$

$$0 < a \leqslant b \;\Rightarrow\; w(a) \geqslant w(b).$$

Example 2.2. We have seen that the natural valuation is compatible with the order. Moreover, K_v is convex.

Proposition 2.3. *(Characterization of compatible valuations).*
The following are equivalent:

(1) *w is compatible with the order of K.*

(2) *K_w is convex.*

(3) *I_w is convex.*

(4) $I_w < 1$.

(5) $1 + I_w \subseteq K^{>0}$.

(6) *The residue map*

$$K_w \to Kw,\, a \mapsto a + I_w$$

induces an ordering on Kw given by

$$a + I_w \geqslant 0 \;:\Leftrightarrow\; a \geqslant 0.$$

(7) *The group*

$$\mathcal{U}_w^{>0} := \{a \in K : w(a) = 0 \;\wedge\; a > 0\}$$

of positive units is a convex subgroup of $(K^{>0}, \cdot, 1, <)$.

Proof. (1) $\Rightarrow$ (2). $0 < a \leqslant b \in K_w \Rightarrow w(a) \geqslant w(b) \geqslant 0 \Rightarrow a \in K_w$.

(2) $\Rightarrow$ (3). Let $a, b \in K$ with $0 < a < b \in I_w$. Since $w(b) > 0$, it follows that $w(b^{-1}) = -w(b) < 0$ and then $b^{-1} \notin K_w$.
Therefore also $a^{-1} \notin K_w$, because $0 < b^{-1} < a^{-1}$ and K_w is convex by assumption. Hence $w(a) > 0$ and $a \in I_w$.

(3) $\Rightarrow$ (4). Otherwise $1 \in I_w$ but $w(1) = 0$, contradiction.

(4) $\Rightarrow$ (5). Clear.

□