REAL ALGEBRAIC GEOMETRY LECTURE NOTES
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1. Real closed fields of power series

Notation 1.1. For $\mathbb{K} = k((G))$ let $k(G)$ denote the subfield of $\mathbb{K}$ generated by $k \cup \{t^g : g \in G\}$.

Theorem 1.2. *Let K be a real closed field, v its natural valuation, $G = v(K^\times)$ its value group, $\overline{K}$ its residue field. Then K is order isomorphic to a subfield $i(K)$ such that*

$$\overline{K}(G)^{\mathrm{rc}} \subseteq i(K) \subseteq \overline{K}((G)).$$

Remark 1.3. We denote by $k(G)^{\mathrm{rc}}$ the relative algebraic closure of $k(G)$ in $\mathbb{K}$. Note that if $\mathbb{K}$ is real closed, then $k(G)^{\mathrm{rc}}$ is (isomorphic to) the real closure of $k(G)$ (i.e. K is "sandwiched" between two real closed fields of power series).

Remark 1.4. Note about $k(G)$:

(*i*) Consider all series in $\mathbb{K}$ which have finite support and denote it by $k[G] := \{s \in \mathbb{K} : \mathrm{support}(s) \text{ is finite}\}$.

ÜB: $k[G]$ is a subring of $\mathbb{K}$, so it is a domain, called the **group ring** over k and the group G.

Excurs about $k[G]$: Let $s \in k[G]$, $\mathrm{support}(s) = \{g_1, \ldots, g_r\}, r \in \mathbb{N}$, i.e. there are coefficients $c_1, \ldots, c_r \in k$ such that $s = c_1 t^{g_1} + \ldots + c_r t^{g_r}$, so the group ring $k[G]$ can be viewed as the ring of "polynomials" with coefficients in k and variables in $\{t^g : g \in G\}$.
Example: If $G = \mathbb{Z}$, say $k = \mathbb{R}$ or $k = \mathbb{C}$, then $k[G]$ is called the **ring of Laurent polynomials**.

(*ii*) $k(G) = \mathrm{ff}(k[G]) = k(t^g : g \in G)$.

2. Embedding of the value group

The aim of this section is to prove that the value group of a real closed field K under its natural valuation can be embedded into the multiplicative subgroup $(K^{>0}, \cdot, 1, <)$.

Proposition 2.1. *Let K be an ordered field and $G = v(K^{\times})$, where v denotes the natural valuation.*

(i) *the map*

$$\nu : (K^{>0}, \cdot, 1, <) \to G,\ a \mapsto -v(a) = v(a^{-1})$$

is a surjective homomorphism of ordered groups with kernel

$$U_v^{>0} = \{a \in K_v : a > 0, v(a) = 0\}.$$

So $U_v^{>0}$ is a convex subgroup of $(K^{>0}, \cdot, 1, <)$ and $K^{>0}/U_v^{>0} \cong G$.

(ii) *if moreover $K^{>0}$ is divisible (in particular this is the case if K is real closed), then $(K^{>0}, \cdot, 1, <) = B \cdot U_v^{>0}$, where B is a multiplicative subgroup of $(K^{>0}, \cdot, 1)$ and is order-isomorphic to G.*

Remark 2.2. Here we are considering $(K^{>0}, \cdot, 1, <)$ as a $\mathbb{Q}$-vector space as follows:

(i) $(K^{>0}, \cdot, 1, <)$ is an ordered abelian group.
(ii) Define the scalar map $\mathbb{Q} \times K^{>0} \to K^{>0}$, $(q, a) \mapsto a^q$.
Note that $U_v^{>0}$ is also divisible. Use the Theorem from LA1 about existence and uniqueness up to isomorphism of a complement to a subspace in a vector space.

Proof. (of the proposition)

(i) Note that

$$\nu(ab) = -v(ab) = -v(a) - v(b) = \nu(a) + \nu(b).$$

To show surjectivity let $g \in G$ and choose $a > 0$, $a \in K$, such that $-v(a) = g$ (then $\nu(a) = g$).
Order-preserving: Let $a \geqslant 1$. Show $\nu(a) \geqslant 0$, i.e. $-v(a) \geqslant 0$ or $v(a) \leqslant v(1)$ (via Archimedean equivalence classes).
Compute kernel:

$$a \in \ker\, \nu \Leftrightarrow \nu(a) = 0 \Leftrightarrow -v(a) = 0 \Leftrightarrow v(a) = 0 \Leftrightarrow a \in U_v^{>0},$$

since $a \in K^{>0}$.

□

Corollary 2.3. *If K is a totally ordered field such that $(K^{>0}, \cdot, 1)$ is divisible (in particular if K is real closed), then there exists an order preserving embedding of $v(K^{\times})$ into $(K^{>0}, \cdot, 1, <)$.*

3. EMBEDDING OF THE RESIDUE FIELD

In this section we prove that the residue field of a real closed field K, with respect to the natural valuation, embedds in K.

Proposition 3.1. *Let K be a real closed field. Then there exists a subfield of K which is order-isomorphic to the residue field $\overline{K}$ of K with respect to the natural valuation (i.e. the residue field embedds in K).*

Proof. We want to apply Zorn's lemma to the collection Θ of all Archimedean subfields of K, which is partially ordered under inclusion. Note that $\mathbb{Q}$ is Archimedean, i.e. Θ is non-empty. Now let $\mathcal{C} \subseteq \Theta$ be a totally ordered subset. We need to find an upper bound in Θ. Set $\mathcal{S} = \bigcup \mathcal{C}$ and verify that this is indeed an upper bound.
Let $k \subseteq K$ be a maximal Archimedean subfield. We will show $k \cong \overline{K}$. Note that $k^\times \subset U_v$. Consider the residue map $k \to \overline{K}$, $x \mapsto \overline{x}$. This is an injective homomorphism. We claim that it is also surjective.
First of all note that k is real closed. This is because the real closure of an Archimedean field is Archimedean. Moreover the real closure of a subfield of K_v is a subfield of K_v. Indeed $v(z) = 0$ for any z in the relative algebraic closure of k, because $v(z)$ is in the divisible hull $\widetilde{v(k)} = \{0\}$ of $v(k)$. So the relative algebraic closure of k, if a proper extension, would contradict the maximal choice of k. Note that by Proposition 4.1 lecture 14, also $\overline{k}$ is real closed.
Now assume the residue map is not surjective, i.e. $\exists \overline{y} \in \overline{K} \backslash \overline{k}$. Let $y \in U_v$ denote a preimage of $\overline{y}$. We claim that $k(y) \subseteq U_v$ and that $(k(y)$ is Archimedean. Note that y is transcendental, so $k(y) = ff(k[y])$. Consider $a_n y^n + \ldots + a_0 \in k[y]$. If

$$\overline{a_n y^n + \ldots + a_0} = \overline{a_n}\,\overline{y}^n + \ldots + \overline{a_0} = 0,$$

then $\overline{y}$ would be algebraic over $\overline{k}$.
So any $z \in k(y)$ has $\overline{z} \neq 0$, so $k(y) \subset U_v$ and is Archimedean (because $\forall z \in k(y) : v(z) = 0$, so $z \sim^+ 1$), contradicting the maximality of k.

□