REAL ALGEBRAIC GEOMETRY LECTURE NOTES (12: 21/05/15 - CORRECTED ON 27/05/2019)

SALMA KUHLMANN

Contents

1. Proof of Neumann's lemma

The aim of today's lecture is to prove Neumann's lemma. By what was shown last time, we then obtain that $k((G))$ is indeed a field.

Proposition 1.1. *Set* $S_n := \operatorname{support}\varepsilon^n$ *and* $S := \bigcup_{n\in\mathbb{N}} S_n$. *Then* S *is a well-ordered set.*

Remark 1.2. *Note that* $\operatorname{support}\varepsilon^n \subseteq \operatorname{support}\varepsilon \oplus \ldots \oplus \operatorname{support}\varepsilon$ *(n-times). Thus,* S_n *is well-ordered for any* $n \in \mathbb{N}$*.*

Proof. (of the proposition)
We argue by contradiction. Let $(u_i : i \in \mathbb{N}) \subseteq S$ be an infinite strictly decreasing sequence. We write

$$u_i = a_{i_1} + \ldots + a_{i_{n_i}},$$

where $a_{i_j} \in S_1 \subset G^{>0}$ $\forall j = 1, \ldots, u_i$. Let v_G denote the natural valuation on G.

ÜB: $\operatorname{sign}(g_1) = \operatorname{sign}(g_2) \Rightarrow v_G(g_1 + g_2) = \min\{v_G(g_1), v_G(g_2)\}$.

Note that $v_G(u_i) = \min\{v_G(a_{i_j})\} \underbrace{=}_{\text{wlog}} v_G(a_{i_1})$. Thus, $v_G(S_u) = v_G(S_1)$. Now recall that

$$0 < g_1 < g_2 \Rightarrow v_G(g_1) \geqslant v_G(g_2).$$

Since $v_G(S_1)$ is anti well-ordered and since $(v_G(u_i) : i \in \mathbb{N}) \subset v_G(S_1)$ is an increasing sequence, it must stabilize after finitely many terms. We assume without loss of generality that it is constant and denote this constant by $U \in v_G(G\backslash\{0\})$, without loss of generality U is as large as possible. So for every $i \in \mathbb{N}$ consider $v_G(u_i) = U = v_G(a_{i_1})$. Let a^* be the smallest element in S_1 for which $v_G(a^*) = U$.

We have that $v_G(u_1) = U = v_G(a^*)$, so $0 < u_1 \leqslant ra^*$ for some $r \in \mathbb{N}$. Fix r. Then $u_i \leqslant ra^*$ $\forall i \in \mathbb{N}$. Since S_1 is well-ordered, it does not contain any infinite decreasing sequence, so we may without loss of generality assume that $n_i > 1\ \forall i \in \mathbb{N}$. We write $u_i = a_{i_1} + v_i$, where $v_i \in S_{n_i-1}$ and $v_i \neq 0\ \forall i$.

Claim: There is a subsequence $(v_{i_k})_k$ of $(v_i)_i$, which is strictly decreasing.

Let us construct this subsequence. Note that the set $\{u_i - v_i : i \in \mathbb{N}\}$ is well-ordered. Proceed as follows:
Let $u_{i_1} - v_{i_1} = \min\{u_i - v_i\}$, let $u_{i_2} - v_{i_2}$ be the smallest element of the set $\{u_i - v_i : i > i_1\}$ etc., so $(u_{i_k} - v_{i_k})_k$ is an increasing sequence, i.e. $u_{i_{k+1}} - v_{i_{k+1}} \geqslant u_{i_k} - v_{i_k}$, so

$$v_{i_{k+1}} - v_{i_k} \leqslant u_{i_{k+1}} - u_{i_k}.$$

Therefore, $(v_{i_k})_k$ is strictly decreasing in S, and this proves the claim.

Now note that $0 < v_i < u_i\ \forall i$. Therefore, $v_G(v_i) \geqslant v_G(u_i) = U$, i.e. $v_G(v_{i_k}) = U\ \forall k$ (recall that U was as large as possible).
But now $a^* \leqslant a_{i_1}$ and $u_i \leqslant ra^*$. Hence,

$$v_i = (u_i - a_{i_1}) \leqslant (r-1)a^*\ \forall i,$$

in particular for all i_k, so $v_{i_k} \leqslant (r-1)a^*\ \forall k$ and $(v_{i_k})_k$ is strictly decreasing with $v_G(v_{i_k}) = U\ \forall k$.

Repeat the argument with the sequence $\{v_{i_k}\} \subset S \subset G^{>0}$ to eventually get a sequence $\leqslant (r-l)a^* < 0$, the desired contradiction.

□

Proposition 1.3. $\forall g \in S : |\{n \in \mathbb{N} : g \in S_n\}| < \infty$.

Proof. Assume $\exists a \in S$ such that $|\{n \in \mathbb{N} : a \in S_n\}| = \infty$. Since S is well-ordered, we may choose a to be the smallest such element of S. Write

$$a = a^j_{i_1} + \ldots + a^j_{i_{n_j}} \in S_{n_j} \quad (*)$$

where n_j is strictly increasing in $\mathbb{N}$ and $a^j_{i_k} \in S_1$. So $\{a^j_{i_1} : j \in \mathbb{N}\} \subseteq S_1$ is well-ordered. Thus, this set has an infinite increasing sequence, assume without loss of generality that $(a^j_{i_1} | j \in \mathbb{N})$ is increasing.

Denote by $a'_j := a^j_{i_2} + \ldots + a^j_{i_{n_i}} \in S_{n_j-1}$, so $a'_j < a\ \forall i \in \mathbb{N}$. Since $(*)$ is constant and $(a_{i_1} | i \in \mathbb{N})$ is increasing, we obtain that $\{a'_j : j \in \mathbb{N}\}$ is decreasing and contained in S. Therefore it stabilizes, i.e. becomes ultimately constant. Denote this constant by $a'_j := a'\ \forall j >> N$. So $a' \in S_{n_j-1}$, and therefore

$$\left|\{n \in \mathbb{N} : a' \in S_n\}\right| = \infty\ \forall j >> N,$$

and $a' < a$ because $a' = a'_j < a\ \forall j >> N$, contradicting the minimality of a.
□

The two propositions finish the proof of Neumann's lemma.