REAL ALGEBRAIC GEOMETRY LECTURE NOTES (10: 11/05/15 - CORRECTED ON 20/05/2019)

SALMA KUHLMANN

Contents

Chapter II: Valuations on ordered fields (particularly real closed fields)

1. Valued fields

Definition 1.1. Let K be a field, G an ordered abelian group and ∞ an element greater than every element of G. A surjective map

$$w\colon K \longrightarrow G \cup \{\infty\}$$

is a **valuation** if and only if $\forall\, a, b \in K$:

(*i*) $w(a) = \infty \;\Leftrightarrow\; a = 0$,

(*ii*) $w(ab) = w(a) + w(b)$,

(*iii*) $w(a-b) \geqslant \min\{w(a), w(b)\}$.

Immediate consequences:

- $w(1) = 0$,
- $w(a) = w(-a)$,
- $w(a^{-1}) = -w(a)$ if $a \neq 0$,
- $w(a) \neq w(b) \;\Rightarrow\; w(a+b) = \min\{w(a), w(b)\}$.

Definition 1.2.

(*i*) $R_w := \{a \in K : w(a) \geqslant 0\}$ is a subring of K, called the **valuation ring** of w.

(*ii*) $I_w := \{a \in K : w(a) > 0\} \subseteq R_w$ is called the **valuation ideal** of w.

(*iii*) $U_w := \{a \in R_w : a^{-1} \in R_w\} = \{a \in R_w : w(a) = 0\}$ is a multiplicative subgroup of R_w and is called the **group of units** of R_w.

Remark 1.3.

- Note that $R_w = U_w \,\dot{\cup}\, I_w$. By this observation one can immediately show that R_w is a local ring with unique maximal ideal I_w.
- Note that for any $x \in K^\times$ either $x \in R_w$ or $x^{-1} \in R_w$ (or both in case $x \in U_w$).

Definition 1.4.

(*i*) The **residue field** is denoted by $K_w := R_w/I_w$.

(*ii*) The **residue map** $R_w \to K_w$, $a \mapsto \overline{a} := aw$ is the canonical projection.

(*iii*) The **group of** 1**-units** of R_w is denoted by

$$1 + I_w := \{a \in R_w : w(a-1) > 0\}$$

and is a multiplicative subgroup of U_w.

2. The natural valuation of an ordered field

Let $(K, +, \cdot, 0, 1, <)$ be an ordered field.

Remark 2.1. $(K, +, 0, <)$ is an ordered divisible abelian group.

So on $(K, +, 0, 1)$ we have already defined the natural valuation, namely via the "Archimedean equivalence relation":

$$\begin{array}{rcl} 0 \neq a & \mapsto & v(a) := [a] \\ 0 & \mapsto & \infty \end{array}$$

We have set $G := (K, +, 0, 1)/\sim^+$ and totally ordered G by

$$[a] < [b] :\Leftrightarrow b <<^+ a.$$

We shall show now that we can endow the totally ordered value set $(G, <)$ with a group operation $+$ such that $(G, +, <)$ is a totally ordered abelian group. For every $a, b \in K \setminus \{0\}$ define

$$[a] + [b] := [ab],$$

or in valuation notation

$$v(a) + v(b) := v(ab).$$

Lemma 2.2.

(*i*) *$(G, +, <)$ is an ordered abelian group.*

(*ii*) *The map $v : (K, +, \cdot, 0, 1, <) \to G \cup \{\infty\}$ is a (field) valuation.*

From now on let K be an ordered field and $v : K \to G \cup \{\infty\}$ its natural valuation, with value group $v(K^*) = G$.

Consider

$R_v := \{a \in K : v(a) \geqslant 0\}$,

$I_v := \{a \in K : v(a) > 0\}$.

What are R_v and I_v (from the point of view of chapter 1)?

$$\begin{aligned} R_v : &= \{a : [a] \geqslant [1]\} \\ &= \{a : a \sim^+ 1 \text{ or } a <<^+ 1\} \\ &= \{a : v(a) \geqslant v(1)\}. \\ I_v : &= \{a : [a] > [1]\} \\ &= \{a : a <<^+ 1\} \\ &= \{a : v(a) > v(1)\}. \end{aligned}$$

Proposition 2.3. *(Properties of the natural valuation)*

(1) *The valuation ring R_v is a convex subring of K. It consists of all the elements of K that are bounded in absolute value by some natural number $n \in \mathbb{N}$. Therefore R_v is often called the ring of bounded elements, or the ring of finite elements.*
This valuation ring of the natural valuation is indeed the convex hull of $\mathbb{Q}$ in K. It is the smallest convex subring of $(K, <)$.

(2) *The valuation ideal I_v is a convex ideal. It consists of all elements of K that are strictly bounded in absolute value by $\frac{1}{n}$ for every $n \in \mathbb{N}$. Therefore I_v is called the ideal of infinitely small elements, or ideal of infinitesimal elements.*

(3) *The residue field K_v is Archimedean, i.e. a subfield of $\mathbb{R}$.*