

REAL ALGEBRAIC GEOMETRY LECTURE NOTES
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1. PSEUDO-COMPLETENESS

In the last lecture we showed that pseudo complete implies maximally valued. Today, we prove the converse implication.

Proposition 1.1. *The Hahn product $(\mathbf{H}_{\gamma \in \Gamma} B(\gamma), v_{\min})$ is pseudo-complete.*

Proof. Let $\{a_\rho\}_{\rho \in \lambda}$ be pseudo-Cauchy. Recall that $\gamma_\rho = v(a_\rho - a_{\rho+1})$ is a strictly increasing sequence. Define $x \in \mathbf{H}_{\gamma \in \Gamma} B(\gamma)$ by

$$x(\gamma) = \begin{cases} a_\rho(\gamma) & \text{if } \gamma < \gamma_\rho \text{ for some } \rho. \\ 0 & \text{otherwise.} \end{cases}$$

This is well-defined because if $\rho_1 < \rho_2 \in \lambda$, $\gamma < \gamma_{\rho_1}$ and $\gamma < \gamma_{\rho_2}$, then $v(a_{\rho_1} - a_{\rho_2}) = \gamma_{\rho_1}$ and therefore

$$a_{\rho_1}(\gamma) = a_{\rho_2}(\gamma)$$

(note that $v_{\min}(a - b)$ is the first spot where a and b differ).

Now we show that $\text{support}(x)$ is well-ordered.

Let $A \subseteq \text{support}(x)$, $A \neq \emptyset$ and $\gamma_0 \in A$. Then $\exists \rho$ such that $\gamma_0 < \gamma_\rho$ and $x(\gamma_0) = a_\rho(\gamma_0)$ with $\gamma_0 \in \text{support}(a_\rho)$. Consider

$$A_0 := \{\gamma \in A : \gamma \leq \gamma_0\}.$$

Note that since $x(\gamma) = a_\rho(\gamma)$ for $\gamma \leq \gamma_0$ it follows that $A_0 \subseteq \text{support}(a_\rho)$ which is well-ordered, so $\min A_0$ exists in A_0 and it is the least element of A .

We conclude by showing that x is a pseudo-limit. By definition of x follows

$$v(x - a_\rho) \geq \gamma_\rho = v(a_{\rho+1} - a_\rho) \quad \forall \rho \in \lambda.$$

If $v(x - a_\rho) > v(a_\rho - a_{\rho+1})$, then

$$v(x - a_{\rho+1}) = v(x - a_\rho + a_\rho - a_{\rho+1}) = v(a_\rho - a_{\rho+1}) = \gamma_\rho,$$

but on the other hand we have

$$v(x - a_{\rho+1}) \geq \gamma_{\rho+1} > \gamma_\rho,$$

a contradiction. □

Corollary 1.2. *Let (V, v) be a valued vector space with $S(V) = [\Gamma, \{B(\gamma), \gamma \in \Gamma\}]$. Then there exists a valuation preserving embedding*

$$(V, v) \hookrightarrow (\mathbb{H}_{\gamma \in \Gamma} B(\gamma), v_{\min})$$

Proof. The picture is the following:

$$\begin{array}{ccc} V'_1 = V \hookrightarrow^{h'} \mathbb{H}_{\gamma \in \Gamma} B(\gamma) = V'_2 & & \\ \text{immediate} \Big\downarrow & & \Big\downarrow \text{immediate} \\ V_1 \xrightarrow[\sim]{h} \bigsqcup_{\gamma \in \Gamma} B(\gamma) = V_2 & & \end{array}$$

Let \mathcal{B} be a maximal valuation independent set in V and set $V_1 = \langle \mathcal{B} \rangle_Q$. Then V_1 has a valuation basis and therefore h exists and $V|_{V_1}$ is immediate. \square

Hilfslemma 1.3. *Let (V_1, v_1) be maximally valued, (V_2, v_2) a valued vector space and $h : V_1 \rightarrow V_2$ a valuation preserving isomorphism. Then (V_2, v_2) is maximally valued.*

Proof. Let $S(V_1) = [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$. Assume that V_2 is not maximally valued, so $\exists V'_2$ a proper immediate extension. By our main theorem there exists an embedding h' of immediate extensions V'_2 into $\mathbb{H}_{\gamma \in \Gamma} B(\gamma)$. This is impossible, since h' cannot be injective. \square

Corollary 1.4. *Let (V, v) be a maximally valued vector space. Then it is pseudo complete. In fact*

$$(V, v) \simeq (\mathbb{H}_{\gamma \in \Gamma} B(\gamma), v_{\min}),$$

where $S(V) = [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$.

Proof. By the first corollary, the picture is the following

$$\begin{array}{ccc} & \mathbb{H} B(\gamma) & \\ & \Big\downarrow \text{immediate} & \\ V \xrightarrow[\sim]{h} & V_2 & \end{array}$$

Since V is maximally valued, it follows from the Hilfslemma that V_2 is maximally valued. Therefore the extension $\mathbb{H} B(\gamma)|_{V_2}$ is not proper, i.e. $V_2 = \mathbb{H} B(\gamma)$. Thus h is surjective, i.e. h is an isomorphism of valued vector spaces $V \rightarrow \mathbb{H}_{\gamma \in \Gamma} B(\gamma)$. \square