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ÜBUNGEN ZUR VORLESUNG POSITIVE POLYNOME

BLATT 10

These exercises will be collected Tuesday 13th July in the mailbox n.14 of the Mathematics department.

Notation 1. Let (X, τ) be a topological space and $A \subseteq X$ a subset. We denote by $\tau_{|A}$ the topology induced on A by τ, namely

$$U \in \tau_{|A} \overset{\text{def}}{\Longleftrightarrow} \exists U' \in \tau \text{ with } U' \cap A = U.$$

Notation 2. Let (X, τ^1), (Y, τ^2) be topological spaces. We denote by $\tau^1 \times \tau^2$ the product topology of τ^1 and τ^2 on $X \times Y$ (we recall that

$$\mathcal{B} = \{ U_1 \times U_2 \mid U_1 \in \tau^1, U_2 \in \tau^2 \}$$

is a basis for $\tau^1 \times \tau^2$).

1. Let (X, τ^1), (Y, τ^2) be topological spaces and $A \subseteq X$, $B \subseteq Y$ subsets. Show that

$$(\tau^1 \times \tau^2)_{|A \times B} \;=\; \tau^1_{|A} \;\times\; \tau^2_{|B},$$

namely that the topology induced on $A \times B$ by the product topology $\tau^1 \times \tau^2$ on $X \times Y$ coincides with the product of the induced topologies on A and on B.

2. Let K be a topological field, V a K-topological vector space and $W \subset V$ be a finite-dimensional subspace.

Show that W is a K-topological vector space with the induced topology from V.

Let $(X, \mathcal{M}, \mu)$ be a **measure space**, namely

- X is a set,
- $\mathcal{M}$ is a σ-algebra in X (the elements in $\mathcal{M}$ are the **measurable sets**),
- $\mu \colon \mathcal{M} \to [0, \infty]$ is a countable additive map (where $[0, \infty]$ stands for $\mathbb{R}_+ \cup \{\infty\}$) .

We recall that a function $f \colon X \to [0, \infty]$ is **measurable** if $f^{-1}(U) \in \mathcal{M}$ for every $U \subset [0, \infty]$ open, where a basis for the topology on $[0, \infty]$ is given by $\{[0, a) \mid a \in \mathbb{R}_+\} \cup \{(a, b) \mid a, b \in \mathbb{R}_+\} \cup \{(a, \infty] \mid a \in \mathbb{R}_+\}$.

χ_A denotes the characteristic function of the set A and a measurable function $s \colon X \to [0, \infty]$ is **simple** if it is of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

for some $\alpha_i \in \mathbb{R}$ and measurable sets $A_i \in \mathcal{M}$.

For every $E \in \mathcal{M}$ and every measurable simple function s as above, we define

$$\int_E s \, d\mu = \sum_{i=1}^{n} \alpha_i \, \mu(A_i \cap E).$$

If $f \colon X \to [0, \infty]$ is measurable and $E \in \mathcal{M}$, we define

$$\int_E f \, d\mu = \sup \int_E s \, d\mu,$$

where s ranges over all measurable simple functions such that $0 \leqslant s \leqslant f$.

3. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions on X, such that

– $0 \leqslant f_n(x) \leqslant f_{n+1}(x)$ for every $n \in \mathbb{N}$ and every $x \in X$;

– $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in X$.

Show that:

(i) f is measurable;

(ii)

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Hints: $0 \leqslant f \leqslant g \Rightarrow \int_X f \, d\mu \leqslant \int_X g \, d\mu$.

Let s be a simple measurable function such that $0 \leqslant s \leqslant f$ and c a constant $0 < c < 1$, and define $E_n = \{x \in X \mid f_n(x) \geqslant cs(x)\}$ $\forall n \in \mathbb{N}$. Observe that

$$\int_X f_n \, d\mu \geqslant c \int_{E_n} s \, d\mu$$

and use it to conclude that

$$\lim_{n \to \infty} \int_X f_n \, d\mu \geqslant \int_X f \, d\mu.$$