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ÜBUNGEN ZUR VORLESUNG POSITIVE POLYNOME

BLATT 06

These exercises will be collected Tuesday 15th June in the mailbox n.14 of the Mathematics department.

1. Let E/F be a field extension. Show that

(*i*) $S \subseteq E$ is algebraically independent over F if and only if
$$\forall\, s \in S : s \text{ is transcendental over } F(S \setminus \{s\}).$$

(*ii*) $S \subseteq E$ is a transcendence base for E/F if and only if S is algebraically independent over F and E is algebraic over $F(S)$.

We recall that a ring is said to be **local** if it contains exactly one maximal ideal.

2. We denote by $\mathbb{R}[[\underline{\mathrm{x}}]]$ the ring of formal power series with coefficients in $\mathbb{R}$.

(*i*) Show that $\mathbb{R}[[\underline{\mathrm{x}}]]$ is a local ring.

(*ii*) Let $f \in \mathbb{R}[[\underline{\mathrm{x}}]]$,
$$f = f_k + f_{k+1} + \dots$$
where every f_i is homogeneous of degree i, $f_k \neq 0$. Assume that f is sos in $\mathbb{R}[[\underline{\mathrm{x}}]]$. Show that k is even and f_k is a sum of squares of forms of degree $k/2$.

3. Consider $K = [-1, 1\,] \subset \mathbb{R}$. Note that $K = K_S = K_{S'}$, where $S, S' \subset \mathbb{R}[\mathrm{x}]$, $S = \{1 - \mathrm{x}, 1 + \mathrm{x}\}$ and $S' = \{1 - \mathrm{x}^2\}$.

(*a*) Show that T_S is saturated.

(*b*) Show that $T_{S'}$ is saturated as well.

4. Let A be a commutative ring with 1 and let $\chi := \mathrm{Hom}(A, \mathbb{R}) = \{\alpha\colon A \to \mathbb{R} \mid \alpha \text{ is a ring homomorphism}\}$. Define the map

$$\begin{array}{rcl} \mathrm{Hom}(A, \mathbb{R}) & \longrightarrow & \operatorname{Sper} A \\ \alpha & \mapsto & P_\alpha := \alpha^{-1}(\mathbb{R}^{\geqslant 0}). \end{array}$$

Show that

(i) the map is well-defined, i.e. $P_\alpha \subseteq A$ is an ordering;

(ii) it is injective, i.e. $\alpha \neq \beta \Rightarrow P_\alpha \neq P_\beta$;

(iii) $\mathrm{support}(P_\alpha) = \ker \alpha$;

(iv) for every $a \in A$ define $\hat{a}\colon \chi \to \mathbb{R}$ by $\hat{a}(\alpha) = \alpha(a)$ and $\mathcal{U}(\hat{a}) := \{\alpha \in \chi \mid \hat{a}(\alpha) > 0\}$; then $\mathcal{B} = \{\, \mathcal{U}(\hat{a}) \mid a \in A\}$ is a pre-base for a topology τ on χ;

(v) for every $a \in A$ the map $\hat{a}\colon \chi \to \mathbb{R}$ is continuous with the respect to the topology τ;

(vi) if τ_1 is another topology on χ such that $\hat{a}$ is continuous for every $a \in A$, then $\mathcal{U}(\hat{a}) \in \tau_1$ for every $\hat{a}$;

(vii) the spectral topology on $\operatorname{Sper} A$ induces (by the map above $\chi \to \operatorname{Sper} A$) a topology on χ which agrees to τ.

5. Let A be a commutative ring with 1 containing $\mathbb{Q}$. Let T be a generating preprime and M a maximal proper T-module. Suppose M is Archimedean. Define the map

$$\begin{array}{rcl} \alpha\colon A & \longrightarrow & \mathbb{R} \\ a & \mapsto & \inf\{r \in \mathbb{Q} : r - a \in M\}. \end{array}$$

Show that α is a ring homomorphism.