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ÜBUNGEN ZUR VORLESUNG POSITIVE POLYNOME

BLATT 01

These exercises will be collected Tuesday 27 April in the mailbox n.14 of the Mathematics department.

1. Let $n \in \mathbb{N}$, $n \geqslant 1$. Consider $\mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ as a vector space over $\mathbb{R}$.

(a) Let $\mathcal{F}_d$ be the subspace of forms of degree d. Find the dimension of $\mathcal{F}_d$.

(b) Let $\mathcal{V}_d$ be the subspace of polynomials of degree $\leqslant d$. Find the dimension of $\mathcal{V}_d$.

2. Let $n \in \mathbb{N}$, $n \geqslant 1$. For $f \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$, let

$$\mathcal{Z}(f) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : f(x_1, \ldots, x_n) = 0\}.$$

Prove that $\mathbb{R}^n \setminus \mathcal{Z}(f)$ is dense in $\mathbb{R}^n$.

Is it still true replacing $\mathbb{R}$ by any real closed field R?

3. Let $n \in \mathbb{N}$, $n \geqslant 1$, $f \in \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$. Show that

$$\forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n \;\; f(x_1, \ldots, x_n) \geqslant 0 \quad \Rightarrow \;\; \deg(f) \text{ is even.}$$

4. Let A be a commutative ring with 1.

We recall that $T \subseteq A$ is a **preordering** of A if

$$T + T \subseteq T, \quad T \cdot T \subseteq T, \quad a^2 \cdot T \subseteq T \quad \forall\, a \in A, \quad 1 \in T.$$

A preordering P is an **ordering** of A if

$$A = -P \cup P \quad \text{and} \quad -P \cap P \text{ is a prime ideal of } A.$$

Let A be the ring of continuous functions $f\colon [0,1] \to \mathbb{R}$.

Find a preordering T and an ordering P of A such that the following conditions are satisfied:

(i) $\sum A^2 \ \subset\ T \ \subset\ P$,

(ii) there are infinitely many preorderings T_i with $\sum A^2 \ \subsetneq\ T_i \ \subsetneq\ T$,

(iii) there are infinitely many preorderings T^i with $T \ \subsetneq\ T^i \ \subsetneq\ P$.