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Real Algebraic Geometry II
Exercise Sheet 13

The exercise sheet will not be collected. Please come to our office hours if you have any questions.

Exercise 1 (Multiplicative Lexicographic Decomposition)
Let $(K, \leq)$ be an ordered field. Assume that the group $(K^{>0}, \cdot, 1 <)$ is divisible. Let v be the natural valuation on K, $\overline{K}$ its residue field, G its value group, $\mathcal{O}$ its valuation ring, $\mathcal{M}$ the maximal ideal of $\mathcal{O}$ and $\mathcal{U} := \mathcal{O} \setminus \mathcal{M}$.

(a) Show that there exists a group complement B of $\mathcal{U}^{>0}$ in $(K^{>0}, \cdot, 1 <)$ and a group complement B' of $1 + \mathcal{M}$ in $(\mathcal{U}^{>0}, \cdot, 1, <)$ such that $(K^{>0}, \cdot, 1 <) = B \coprod B' \coprod (1 + \mathcal{M}, \cdot, 1, <)$.

(b) Show that every group complement B of $\mathcal{U}^{>0}$ in $(K^{>0}, \cdot, 1 <)$ is order isomorphic to G through the isomorphism $-v$.

(c) Show that every group complement B' of $1 + \mathcal{M}$ in $(\mathcal{U}^{>0}, \cdot, 1, <)$ is order isomorphic to $(\overline{K}^{>0}, \cdot, 1, <)$.

Definition: Let $(K, \leq)$ be an ordered field and v its non-trivial natural valuation. Let $P_K := K^{>0} \setminus R_v$.
Let $\dot\sim$ be defined by $x \dot\sim y$ if and only if $r \cdot |x| < |y|$ for all $r \in R_v$ and $r \cdot |y| < |x|$ for all $r \in R_v$ $(x, y \in P_K)$.
$\dot\sim$ denotes the multiplicative equivalence relation on P_K.
Denote the class of a by $[a]^{\cdot}$ and call it the multiplicative class of a.

Exercise 2

Let $(K, \leq)$ be an ordered field. Let v be the non-trivial natural valuation of $(K, \leq)$ and R_v its valuation ring. Let w be a valuation on K with valuation ring R_w such that $R_w \neq R_v$ is a convex subring. Let G be the value group of v. Let $G_w := \{v(x) \in G \mid x \in K,\ w(x) = 0\}$. Let $a \in P_K := K^{>0} \setminus R_v$.
Show that the following are equivalent

(i) R_w is principal convex generated by a.

(ii) The class $[a]^{\cdot}$ is the largest of all classes of R_w.

(iii) $[a]^{\cdot}$ is a final segment in R_w.

(iv) G_w is principal with smallest archimedean class $[v(a)]$.

(v) $v_G(G_w) = \Gamma_w$ is a principal final segment with smallest element $v_G\big(v(a)\big)$. (where v_G is the natural valuation on G).

Exercise 3

Let Γ_1 and Γ_2 be a totally ordered set.

(a) Show that if $\Gamma_1 \cong \Gamma_2$, then $\Gamma_1^{\mathsf{sf}} \cong \Gamma_2^{\mathsf{sf}}$.

(b) Compute the order type of $\mathbb{Q}^{\mathsf{fs}}$.