**Universität Konstanz**

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# Real Algebraic Geometry II
## Exercise Sheet 10

**Definition:** An ordered field $K$ is *root closed for positive elements*, if for every $n \in \mathbb{N}$ and every $x \in K^{>0}$ there exists $y \in K$ such that $y^n = x$.

**Definition:** Let $K$ be an ordered field and $v$ the natural valuation on $K$.
$S \subseteq K^{>0}$ is called *value group section*, if there exists an order preserving embedding $t : v(K^\times) \hookrightarrow K^{>0}$ with $v\big(t(g)\big) = g$ for all $g \in v(K^\times)$, such that $S = t\big(v(K^\times)\big)$.

**Definition:** Let $K$ be an ordered field, $v$ the natural valuation on $K$ and $k$ the residue field of $(K, v)$. $T \subseteq K$ is called *residue field section*, if there exists an order preserving embedding $\iota : k \hookrightarrow K$ with $\overline{\iota(c)} = c$ for all $c \in K$, such that $T = \iota(k)$.

### Exercise 1

Let $K$ be an ordered field. Let $K$ be root closed for positive elements. Let $v$ be the natural valuation on $K$.

(a) Show that $(K, v)$ admits a value group section.

(b) Assume in addition that $K$ is real closed. Show that $K$ admits a residue field section.

**Definition:** Let $K$ and $L$ be fields. We extend addition and multiplication from $L$ to $L \cup \{\infty\}$ as follows.
For $a \in L$ and $b \in L \setminus \{0\}$

$$\begin{aligned} \infty + a := \quad & a + \infty \quad := \infty \\ \infty \cdot b := \quad & b \cdot \infty \quad := \infty \\ \infty \cdot \infty := \quad & \infty \end{aligned}$$

We will not define $\infty + \infty$, $\infty \cdot 0$ and $0 \cdot \infty$.

A map

$$\varphi : K \longrightarrow L \cup \{\infty\}$$

is called *place* (German *Stelle*) on $K$, if for all $x$, $y \in K$ the following is true whenever the right side is defined:

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$

(ii) $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$

(iii) $\varphi(1) = 1$.

**Exercise 2**

Let $K$ be a field.

(a) Assume $L$ is a field and $\varphi : K \longrightarrow L \cup \{\infty\}$ is a place on $K$. Show that

$$\mathcal{O} := \varphi^{-1}(L)$$

is a valuation ring on $K$ with maximal ideal

$$\mathcal{M} = \varphi^{-1}(\{0\})$$

and residue field

$$\mathcal{O}/\mathcal{M} \cong \varphi(\mathcal{O}).$$

(b) Let $\mathcal{O}$ be a valuation ring on $K$ with maximal ideal $\mathcal{M}$. Let

$$\widetilde{\varphi} : \mathcal{O} \twoheadrightarrow \mathcal{O}/\mathcal{M}$$

be the residue map of $(K, \mathcal{O})$. Define

$$\varphi : K \longrightarrow \mathcal{O}/\mathcal{M} \cup \{\infty\}$$

by

$$\varphi(x) := \begin{cases} \widetilde{\varphi}(x), & x \in \mathcal{O} \\ \infty \quad , & x \notin \mathcal{O} \end{cases}$$

for $x \in K$.

Show that $\varphi$ is a place on $K$.

**Definition:** Let $(K, \mathcal{O})$ be a valued field. The place as in Exercise 2 (b) is called the canonical place of $\mathcal{O}$.

**Exercise 3**

Let $(K, \leq)$ be an ordered field. Let $v$ be a valuation on $K$ with valuation ring $\mathcal{O}$, maximal ideal $\mathcal{M}$ and residue field $\overline{K}$. Let $\varphi : \mathcal{O} \to \overline{K}$ be the canonical place of $\mathcal{O}$.

Show that the following are equivalent

(i) If $a, b \in K$ with $0 < a < b$, then $v(a) \geq v(b)$.

(ii) $\mathcal{M}$ is convex in $(K, \leq)$.

(iii) $\mathcal{O}$ is convex in $(K, \leq)$.

(iv) $\overline{P} := \{p + \mathcal{M} \mid p \in K^{\geq 0} \cap \mathcal{O}\}$ is an ordering on $\overline{K}$.

(v) If $x, y \in \mathcal{O}$ with $x \leq y$, then $\varphi(y) - \varphi(x) \in \overline{P}$.

(vi) $1 + \mathcal{M} \subseteq K^{>0}$.

(vii) If $x \in \mathcal{M}$, then $|x| < q$ for all $q \in \mathbb{Q}$ (where $|\,.\,|$ is the absolute value induced by $\leq$).

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The exercise will be collected **Thursday, 25/06/2015** until 10.00 at box 13 near F 441.

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