REAL ALGEBRAIC GEOMETRY LECTURE NOTES (23: 06/07/15)

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Appendix

The goal of this lecture is to describe the real closure of a Hardy field. In particular, we want to prove the following theorem:

Theorem 0.1. *(Main Theorem)*
The real closure of a Hardy field is again a Hardy field.

1. Preliminaries

Notation 1.1.

- If f is a differentiable function from some half-line (a, ∞) to $\mathbb{C}$, we will denote by $\delta(f)$ the derivative of f.
- If k is a field and $P \in k[X]$, let P' denote the derivative of P and $Z(P)$ the set of roots of P.
- $F := \{f : (a, \infty) \to \mathbb{C} \mid a \in \mathbb{R}\}$.
- $G := \{f : (a, \infty) \to \mathbb{R} \mid a \in \mathbb{R}\} \subseteq F$.
- For $f, g \in F$ define
$$f \sim g :\Leftrightarrow \exists a \in \mathbb{R}\, \forall x > a : f(x) = g(x).$$
Then $\sim$ is an equivalence relation on F. Denote by $\overline{f}$ the equivalence class of f.
- Denote $\mathcal{F} := F/\sim$ and $\mathcal{G} := G/\sim$. Then $\mathcal{F}$ and $\mathcal{G}$ are rings with operations defined by:
$$\overline{f} + \overline{g} = \overline{f+g} \text{ and } \overline{f}\,\overline{g} = \overline{fg}.$$
- We say that $\overline{f}$ is differentiable if there exists $a \in \mathbb{R}$ such that f is differentiable on (a, ∞), and in that case we define the derivative of $\overline{f}$ as $\delta(\overline{f}) := \overline{\delta(f)}$

Definition 1.2.

(*i*) A **Hardy field** is a subring K of $\mathcal{G}$ which is a field and such that for every $\overline{f} \in K$, $\overline{f}$ is differentiable and $\delta(\overline{f}) \in K$.

(*ii*) A **complex Hardy field** is a subring K of $\mathcal{F}$ which is a field and such that for every $\overline{f} \in K$, $\overline{f}$ is differentiable and $\delta(\overline{f}) \in K$.

Definition 1.3. Let K be a Hardy field and $P \in K[X]$ of degree n, say $P = \sum_{m=0}^{n} \overline{f}_m X^m$. If $a \in \mathbb{R}$ is such that $f_1, \ldots, f_n$ are all defined and C^1 on (a, ∞) and $f_n(x) \neq 0$ for all $x > a$, we say that P is **defined** on (a, ∞). Note that such an a always exists.

Notation 1.4. If P is defined on (a, ∞), then for any $x > a$ we define $P_x := \sum_{m=0}^{n} f_m(x) X^m \in \mathbb{R}[X]$.

Remark 1.5. Note that P_x also has degree n and that $(P_x)' = (P')_x$, which we will just denote by P'_x. Of course, the definition of P_x depends on the choice of representatives for $\overline{f}_1, \ldots, \overline{f}_n$. However, whenever a polynomial is introduced, we will always assume we have fixed the representatives of its coefficients, so that P_x is well-defined.

Remark 1.6. Note that if $g \in F$, then $P(\overline{g})$ is the germ of the function $\sum f_i g^i$, so $P(\overline{g}) = 0$ if and only if there exists some a such that $P_x(g(x)) = 0$ for all $x > a$.

Recall 1.7. Let K be a field and $P \in K[X]$.

(*i*) P has only simple roots in its splitting field iff $gcd(P, P') = 1$ iff there exist $A, B \in K[X]$ such that $AP + BP' = 1$.

(*ii*) If $char(K) = 0$ and P is irreducible, then $gcd(P, P') = 1$.

The keystone of the proof of the main theorem is a well-known theorem from analysis, namely the **implicit function theorem**, which we recall here.

Theorem 1.8. *(IFT)*
Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open, $u : U \times V \to \mathbb{R}^m$ a C^k function for some $k \in \mathbb{N}$ and $(x_0, y_0) \in U \times V$ such that $u(x_0, y_0) = 0$ and $det(\frac{\partial u}{\partial y}(x_0, y_0)) \neq 0$. Then there exists an open ball U_0 containing x_0, an open ball V_0 containing y_0 and a C^k function $\phi : U_0 \to V_0$ such that for any $(x, y) \in U_0 \times V_0$:

$$u(x, y) = 0 \Leftrightarrow y = \phi(x).$$

We will actually need a particular form of the implicit function theorem, namely:

Theorem 1.9. *(IFT')*
Let K be a Hardy field, $P \in K[X]$ defined on (a, ∞), $x_0 > a$ and y_0 a complex root of P_{x_0} which is not a root of P'_{x_0}. Then there exists an open interval I containing x_0, an open ball U containing y_0 and a C^1 function $\phi : I \to U$ such that:

$$(*) \quad \forall (x, y) \in I \times U : P_x(y) = 0 \Leftrightarrow y = \phi(x)$$

Proof. Set

$$u : (a, \infty) \times \mathbb{C} \to \mathbb{C},\ (x, y) \mapsto P_x(y).$$

Then u is C^1 on $(a, \infty) \times \mathbb{C}$. By assumption, we have $u(x_0, y_0) = 0$ and $\frac{\partial u}{\partial y}(x_0, y_0) = P'_{x_0}(y_0) \neq 0$, so we can apply the IFT to the function u at the point (x_0, y_0). □

2. Proof of the Main Theorem

Lemma 2.1. *Let K be a Hardy field and $P \in K[X]$ defined on (a, ∞). If $gcd(P, P') = 1$, then there exists some $b > a$ such that $gcd(P_x, P'_x) = 1$ for all $x > b$.*

Proof. Since $gcd(P, P') = 1$, there are $A, B \in K[X]$ such that $AP + BP' = 1$. Now let $b > a$ such that A, B are defined on (b, ∞); for $x > b$ we have $A_x P_x + B_x P'_x = 1$, hence $gcd(P_x, P'_x) = 1$. □

Lemma 2.2. *Let K be a Hardy field, $P \in K[X]$ non-zero defined on (a, ∞) and f a continuous function from (a, ∞) to $\mathbb{C}$ such that $P_x(f(x)) = 0$ and $P'_x(f(x)) \neq 0$ for all $x > a$. Then f is differentiable on (a, ∞).*

Proof. Let $x_0 > a$, $y_0 := f(x_0)$. By hypothesis, y_0 is a root of P_{x_0} but not of P'_{x_0}. Thus, we may apply IFT', and obtain I, U and ϕ as in IFT' such that $(*)$ holds.
Set $J := I \cap f^{-1}(U)$. U is a neighborhood of y_0 and f is continuous, so $f^{-1}(U)$ is a neighborhood of x_0, so J is also a neighborhood of x_0. Let $x \in J$; by assumption we have $P_x(f(x)) = 0$ and $(x, f(x)) \in I \times U$, which by $(*)$ implies that $f(x) = \phi(x)$.
Therefore $f_{|J} = \phi_{|J}$, which, since ϕ is C^1, implies that f is differentiable at x_0. Since x_0 was chosen arbitrarily, we obtain that f is differentiable on (a, ∞). □

Proposition 2.3. *Let K be a Hardy field and $f \in F$ a continuous function such that there exists $P \in K[X]$ non-zero such that $P(\overline{f}) = 0$. Then the ring $K[\overline{f}]$ is a complex Hardy field. If f happens to be in G, then $K[\overline{f}]$ is a Hardy field.*

Proof. Without loss of generality we can assume that P is irreducible. This implies that $K[\overline{f}]$ is isomorphic to $K[X]/(PK[X])$, so it is a field. We now have to show that every element of $K[\overline{f}]$ is differentiable and that $K[\overline{f}]$ is stable under derivation. It is sufficient to show that $\overline{f}$ is differentiable and that $\delta(\overline{f}) \in K[\overline{f}]$.

Since $P(\overline{f}) = 0$, there exists some $a \in \mathbb{R}$ such that $P_x(f(x)) = 0$ for all $x > a$. As P is irreducible and $\mathrm{char}(K) = 0$, $gcd(P, P') = 1$, so by Lemma 2.1 there exists some $b > a$ such that $gcd(P_x, P'_x) = 1$ for all $x > b$. Hence, P_x and P'_x have no root in common. Thus, $P_x(f(x)) = 0 \neq P'_x(f(x))$ for any $x > b$. Now apply Lemma 2.2 and obtain that f is differentiable on (b, ∞). Set $P = \sum_{m=0}^{n} \overline{g}_m X^m$. Then

$$\begin{aligned} 0 = \delta(P(\overline{f})) &= \sum_{m=0}^{n} \delta(\overline{g}_m \overline{f}^m) \\ &= \delta(\overline{g_0}) + \sum_{m=1}^{n} (\delta(\overline{g}_m)\overline{f}^m + m\overline{g}_m \overline{f}^{m-1}\delta(\overline{f})) \\ &= \sum_{m=0}^{n} \overline{\delta(g_m)}\,\overline{f}^m + \delta(\overline{f}) \sum_{m=1}^{n} m\overline{g}_m \overline{f}^{m-1} \\ &= Q(\overline{f}) + \delta(\overline{f})P'(\overline{f}) \end{aligned}$$

with $Q \in K[X]$, hence $\delta(\overline{f}) = \frac{-Q(\overline{f})}{P'(\overline{f})} \in K(\overline{f}) = K[\overline{f}]$. □

Lemma 2.4. *Let K be a Hardy field, $n \in \mathbb{N}$ and $P \in K[X]$ of degree n defined on (a, ∞), such that P_x has n distinct roots in $\mathbb{C}$ for all $x > a$. For any pair $(x_0, y_0) \in (a, \infty) \times \mathbb{C}$ such that y_0 is a root of P_{x_0}, there exists a C^1 function $\phi : (a, \infty) \to \mathbb{C}$ such that $y_0 = \phi(x_0)$ and*

$$\forall x > a : P_x(\phi(x)) = 0 \quad (\dagger)$$

Proof. Let $x_0 > a$ and y_0 a complex root of P_{x_0}. Since P_{x_0} has simple roots, y_0 is not a root of P'_{x_0}, so we can apply IFT' and we get an open interval I containing x_0, an open ball U containing y_0 and a C^1 function $\phi : I \to U$ such that $(*)$ is satisfied, which in particular implies that $\phi(x_0) = y_0$ and $P_x(\phi(x)) = 0$ for all $x \in I$. Define $\mathcal{E}$ to be the set

$$\{(J, \psi) \mid I \subseteq J \text{ open interval, } \psi\ C^1\text{-extension of } \phi \text{ to } J \text{ satisfying } (\dagger) \text{ on } J\}.$$

Note that $\mathcal{E}$ is non-empty since $(I, \phi) \in \mathcal{E}$. We can partially order $\mathcal{E}$ by saying that $(J, \psi) \leqslant (J', \chi)$ if $J \subseteq J'$ and χ extends ψ.
Let $(J_h, \psi_h)_{h \in H}$ be a chain in $\mathcal{E}$. Set $J := \bigcup_{h \in H} J_h$ and define ψ on J by $\psi(x) = \psi_h(x)$ if $x \in J_h$; this is well-defined because ψ_h is an extension of $\psi_{h'}$ for any $h, h' \in H$ such that $J_{h'} \subseteq J_h$. If $x \in J$, then $x \in J_h$ for some $h \in H$, and since $(J_h, \psi_h) \in \mathcal{E}$ we have $P_x(\psi_h(x)) = 0$, hence $P_x(\psi(x)) = 0$. Thus, ψ satisfies $(\dagger)$ on J, so $(J, \psi) \in \mathcal{E}$. Moreover, we have $(J_h, \psi_h) \leqslant (J, \psi)$ for any $h \in H$, so (J, ψ) is an upper bound of $(J_h, \psi_h)_{h \in H}$.
We just proved that any chain of $\mathcal{E}$ has an upper bound. By Zorn's lemma, it follows that $\mathcal{E}$ has a maximal element (J, ψ)

To conclude the proof, we have to show that $J = (a, \infty)$. Set $b := \sup J$. Towards a contradiciton, assume that $b \neq \infty$. By hypothesis, P_b has n distinct roots $y_1, \ldots, y_n$, none of which is a root of P'_b. We apply IFT' at each of the points $(b, y_1), \ldots, (b, y_n)$, and we obtain open intervals $I_1, \ldots, I_n$ containing b, open balls $U_1, \ldots, U_n$ containing $y_1, \ldots, y_n$ and C^1 functions $\phi_1 : I_1 \to U_1, \ldots, \phi_n : I_n \to U_n$, such that for each $m \in \{1, \ldots, n\}$, for any

$(x,y) \in I_m \times U_m$, $P_x(y) = 0 \Leftrightarrow y = \phi_m(x)$. Since $y_1, \dots, y_n$ are pairwise distinct,we can choose the sets $U_1, \dots, U_n$ so small that they are pairwise disjoint.

Let $I' := \bigcap_{m=1}^n I_m$. For any $x \in I'$, we have $\phi_1(x) \in U_1, \dots, \phi_n(x) \in U_n$; since $U_1, \dots, U_n$ are pairwise disjoint, $\phi_1(x), \dots, \phi_n(x)$ are pairwise distinct. By $(*)$, each $\phi_m(x)$ is a root of P_x; since P_x has n roots, it follows that $Z(P_x) = \{\phi_1(x), \dots, \phi_n(x)\} \subseteq \bigcup_{m=1}^n U_m$.

Let $J' := I' \cap J$; note that J' is an interval. For any $x \in J'$, $(\dagger)$ implies that $\psi(x)$ is a root of P_x, hence $\psi(x) \in \bigcup_{m=1}^n U_m$. Thus, $\psi(J') \subseteq \bigcup_{m=1}^n U_m$. Since ψ is continuous, $\psi(J')$ is connected. Since $U_1, \dots, U_n$ are pairwise disjoint, this implies that there exists $m \in \{1, \dots, n\}$ such that $\psi(J') \subset U_m$.

Let $x \in J'$; we have $(x, \psi(x)) \in I_m \times U_m$ and $P_x(\psi(x)) = 0$. Since ϕ_m satisfies $(*)$ on $I_m \times U_m$, it follows that $\psi(x) = \phi_m(x)$. This proves that $\psi_{|J'} = \phi_{m|J'}$.

Define the function $\tilde{\psi}$ on $J \cup I'$ by $\tilde{\psi}(x) := \begin{cases} \psi(x) & \text{if } x \in J \\ \phi_m(x) & \text{if } x \in I' \end{cases}$.

This definition makes sense because ψ and ϕ_m agree on I'. $\tilde{\psi}$ is a strict extension of ψ. Since ψ and ϕ_m are C^1, $\tilde{\psi}$ is also C^1. Since ψ satisfies $(\dagger)$ on J and ϕ_m satisfies $(*)$ on I', it follows that $\tilde{\psi}$ satisfies $(\dagger)$ on $J \cup I'$, which contradicts the maximality of (J, ψ). Thus, $b = \infty$ (note that we could prove the same way that $\inf J = a$). □

Lemma 2.5. *Let K be a Hardy field and $P \in K[X]$ of degree n such that $gcd(P, P') = 1$. Then there exists some $a \in \mathbb{R}$ and n C^1 functions $\phi_1, \dots \phi_n : (a, \infty) \to \mathbb{C}$ such that $Z(P_x) = \{\phi_1(x), \dots, \phi_n(x)\}$ for each $x > a$.*

Proof. By Lemma 2.1, there exists some $a_0 \in \mathbb{R}$ such that $gcd(P_x, P'_x) = 1$ for all $x > a_0$, which means that P_x has n distinct roots in $\mathbb{C}$. Let $a > a_0$, and let $y_1, \dots, y_n$ be the n distinct roots of P_a. By the previous lemma, we obtain n C^1 functions $\phi_1, \dots, \phi_n : (a_0, \infty) \to \mathbb{C}$ such that $\phi_m(a) = y_m$ for any $m \in \{1, \dots, n\}$, and $\{\phi_1(x) \dots, \phi_n(x)\} \subseteq Z(P_x)$ for any $x > a$. To show equality, we just have to show that $\phi_l(x) \neq \phi_m(x)$ for any $x > a$ and any $m, l \in \{1, \dots, n\}$.

Now let $m, l \in \{1 \dots n\}$ and $E := [a, \infty] \cap (\phi_m - \phi_l)^{-1}(\{0\})$. Assume $E \neq \varnothing$. By continuity of ϕ_m and ϕ_l, E is a closed subset of $\mathbb{R}$ and has a lower bound a, so it has a minimum b. Since $\phi_m(a) \neq \phi_l(a)$, $b > a$. Set $c := \phi_m(b)$. c is a root of P_b, so we can apply IFT' at the point (b, c) and we get an open neighborhood $I \times U$ of (b, c) and a map $\phi : I \to U$ satisfying $(*)$. Since U is a neighborhood of c, and since $c = \phi_m(b) = \phi_l(b)$, $\phi_l^{-1}(U)$ and $\phi_m^{-1}(U)$ are neighborhoods of b, so

$$J := I \cap (a, \infty) \cap \phi_l^{-1}(U) \cap \phi_m^{-1}(U)$$

is a neighborhood of b. Let $x \in J$ such that $x < b$; $(x, \phi_l(x))$ and $(x, \phi_m(x))$ both belong to $I \times U$ and we have $P_x(\phi_m(x)) = P_x(\phi_l(x)) = 0$; since ϕ satisfies $(*)$ on $I \times U$, this implies $\phi_l(x) = \phi(x) = \phi_m(x)$, so $x \in E$, which contradicts the minimality of b. Thus, $E = \varnothing$. □

Proposition 2.6. *Let k be a Hardy field,*

$$K := \{\overline{f} \in \mathcal{G} \mid \ f \text{ continuous and } \exists P \in k[X] \text{ with } P \neq 0 \wedge P(\overline{f}) = 0\}$$

and

$$L := \{\overline{f} \in \mathcal{F} \mid \ f \text{ continuous and } \exists P \in k[X] \text{ with } P \neq 0 \wedge P(\overline{f}) = 0\}.$$

Then K is a Hardy field, L is a complex Hardy field, L is the algebraic closure of k and K is the real closure of k.

Proof. Obviously, $k \subseteq K \subseteq L$. Now let $\overline{f}, \overline{g} \in K$. By Proposition 2.3, $k[\overline{f}]$ is a Hardy field. Since g is continuous and $\overline{g}$ is canceled by a polynomial in $k[\overline{f}][X]$, we can once again use Proposition 2.3 and we obtain that $k[\overline{f}, \overline{g}]$ is a Hardy field, and since it is algebraic over k, it is contained in K. Since $k[\overline{f}, \overline{g}]$ is a Hardy field, we have

$$0, 1, \overline{f} - \overline{g}, \frac{\overline{f}}{\overline{g}}, \delta(\overline{f}), \delta(\overline{g}) \in k[\overline{f}, \overline{g}],$$

hence

$$0, 1, \overline{f} - \overline{g}, \frac{\overline{f}}{\overline{g}}, \delta(\overline{f}), \delta(\overline{g}) \in K.$$

This proves that K is Hardy field. The same proof shows that L is a complex Hardy field.

Now let us show that L is algebraically closed. Let $P \in k[x]$ irreducible of degree $n > 1$. Since $char(k) = 0$, $gcd(P, P') = 1$. By Lemma 2.5 there exists some $a \in \mathbb{R}$ and C^1 functions $\phi_1, \ldots, \phi_n : (a, \infty) \to \mathbb{C}$, such that for any $x > a$, $Z(P_x) = \{\phi_1(x), \ldots, \phi_n(x)\}$. This means that $\overline{\phi}_1, \ldots, \overline{\phi}_n$ are n distinct roots of P. Since $\phi_1, \ldots, \phi_n$ are continuous functions from (a, ∞) to $\mathbb{C}$ and $\overline{\phi}_1, \ldots, \overline{\phi}_n$ are canceled by $P \in k[X]$, we have $\overline{\phi}_1, \ldots \overline{\phi}_n \in L$.

Thus, any polynomial with coefficients in k splits in L. Since L/k is an algebraic extension, this proves that L is algebraically closed, and thus L is the algebraic closure of k. Finally note that $L = K(i)$. Since $K(i)$ is algebraically closed, K is real closed, and it is the real closure of k. □

Corollary 2.7. *The real closure of a Hardy field is again a Hardy field.*