REAL ALGEBRAIC GEOMETRY LECTURE NOTES (11: 18/05/15)

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Contents

1. The field of generalized power series

Let $k \subseteq \mathbb{R}$ be an Archimedean field and G an ordered abelian group. Recall that we have defined a (totally) ordered abelian group, namely the Hahn product

$$\mathbb{K} := \mathrm{H}_G(k, +, 0, <),$$

i.e. take the Hahn product over the family $S := [G, \{k : g \in G\}]$ with the lexicographic ordering, i.e.

$$\mathbb{K} := \{s : G \to k : \operatorname{support} s \text{ is well-ordered in } G\},$$

where $\operatorname{support} s := \{g \in G : s(g) \neq 0\}$.
Endow this set with pointwise addition of functions, i.e. $\forall s, r \in \mathbb{K}$

$$(s + r)(g) := s(g) + r(g) \in k,$$

and the lexicographic order:

$$s > 0 :\Leftrightarrow s(\min \operatorname{support}(s)) > 0 \text{ in } k \quad \forall s \in \mathbb{K} \backslash \{0\}.$$

We have verified that $(K, +, <_{\text{lex}})$ is an ordered abelian group. Our first goal of today is to make $\mathbb{K}$ into a (totally) ordered field. We need to define multiplication.

Notation 1.1. For $s \in \mathbb{K}$ write

$$s = \sum_{g \in G} s(g) t^g = \sum_{g \,\in\, \operatorname{support} s} s(g) t^g.$$

Definition 1.2. For $r, s \in \mathbb{K}$ define

$$(rs)(g) := \sum_{h \in G} r(g - h) s(h),$$

i.e.

$$sr = \sum_{g \in G} \left(\sum_{h \in G} r(g - h) s(h) \right) t^g.$$

We now adress the following problem: Let $\mathfrak{F} := \{s_i : i \in I\} \subseteq \mathbb{K}$. Can we "make sense" of $\sum_{i\in I} s_i$ as an element of $\mathbb{K}$?

Definition 1.3.

(i) The family $\mathfrak{F}$ is said to be **summable**, if
  (1) $\operatorname{support}\mathfrak{F} := \bigcup_{i\in I} \operatorname{support} s_i$ is well-ordered in G,
  (2) $\forall g \in \operatorname{support}\mathfrak{F}$, the set $S_g := \{i \in I : g \in \operatorname{support} s_i\}$ is finite.

(ii) Assume that $\mathfrak{F}$ is summable. Write

$$\sum_{i\in I} s_i := \sum_{g\,\in\,\operatorname{support}\mathfrak{F}} \left(\sum_{i\in S_g} s_i(g)\right) t^g.$$

We now prove that this multiplication is well-defined. For $h \in G$ define

$$\begin{aligned}\rho_h := t^h r :&= \sum_{g\in G} r(g)t^{g+h}\\ &= \sum_{g\,\in\,\operatorname{support} r} r(g)t^{g+h},\end{aligned}$$

i.e. $\rho_h(g) = r(g-h)\ \forall g \in G$. Note that $\rho_h \in \mathbb{K}$ because

$$\operatorname{support}\rho_h = \operatorname{support} r \oplus \{h\} = \{g+h : g \in \operatorname{support} r\},$$

which is again well-ordered (ÜA).

We now consider

$$\mathfrak{F} := \{s(h)\rho_h : h \in \operatorname{support} s\}.$$

Lemma 1.4. *$\mathfrak{F}$ is summable.*

Note that once the lemma is established we define

$$sr = \sum_{h\,\in\,\operatorname{support} s} s(h)\rho_h = \sum_{g\,\in\,\operatorname{support}\mathfrak{F}} \left(\sum_{h\in S_g} s(h)\rho_h(g)\right) t^g,$$

and comparing, we see that this is the product.

Proof. (1) Show that $\operatorname{support}\mathfrak{F} = \bigcup_{h\in\operatorname{support} s} \operatorname{support}(\rho_h(s(h)))$ is well-ordered. Indeed

$$\begin{aligned}\bigcup_{h\,\in\,\operatorname{support} s} \operatorname{support}(\rho_h s(h)) &= \bigcup_{h\,\in\,\operatorname{support} s} (\operatorname{support} r \oplus \{h\})\\ &= \operatorname{support} s + \operatorname{support} r.\end{aligned}$$

ÜA: If A, B are well-ordered, then $A \oplus B$ is well-ordered.

(2) Show that $S_g = \{h \in \operatorname{support} s : g \in \operatorname{support}(\rho_h s(h))\}$ is finite for $g \in \operatorname{support}\mathfrak{F}$. . We have

$$\begin{aligned}S_g :&= \{h \in \operatorname{support} s : g \in \operatorname{support} r \oplus \{h\}\}\\ &= \{h \in \operatorname{support} s : g = g' + h, g' \in \operatorname{support} r\}\\ &= \{h \in \operatorname{support} s : g - h \in \operatorname{support} r\}.\end{aligned}$$

Assume S_g is infinite. Since S_g is well-ordered, take an infinite strictly increasing sequence in it, say a sequence of $h's$ in it. But then $g - h's$ is an infinite strictly decreasing sequence in support r, contradicting that support r is well-ordered. □

Note we have shown that $\operatorname{support}(rs) \subseteq \operatorname{support} r \oplus \operatorname{support} s$.

Notation 1.5. $\mathbb{K} = k((G))$.

Our next goal is to show that $k((G))$ with the convolution multiplication is a field. We give two proofs:

(1) Follows from "Neumann's lemma" (now)
(2) From S. Prieß-Crampe: $k((G))$ is pseudo-complete (later)

Lemma 1.6. *(Neumann's lemma)*
Let $\varepsilon \in k((G))$ *such that* $\operatorname{support} \varepsilon \subseteq G^{>0}$ (written $\varepsilon \in k((G^{>0}))$) *and* $\{c_n\}_{n\in\mathbb{N}} \subset k^*$. *Then the family* $\mathfrak{F} = \{c_n \varepsilon^n : n \in \mathbb{N}\}$ *is summable, i.e.* $\sum_{n\in\mathbb{N}} c_n \varepsilon^n \in k((G))$.

Corollary 1.7. $k((G))$ *is a field.*

Proof. Let $s \in k((G)), s \neq 0$. Set $g_0 := \min \operatorname{support} s$ and $c_0 = s(g_0) \neq 0$. Write

$$s = c_0 t^{g_0}(1 - \varepsilon),$$

where

$$\varepsilon = - \sum_{\substack{g > g_0 \\ g \,\in\, \operatorname{support} s}} \frac{s(g)}{c_0} t^{g - g_0} \in k((G^{>0})),$$

so

$$s^{-1} := c_0^{-1} t^{-g_0} \left(\sum_{i=0}^{\infty} \varepsilon^i \right).$$

Verify that

$$\left(\sum_{i=0}^{\infty} \varepsilon^i \right)(1 - \varepsilon) = 1,$$

i.e.

$$(1 - \varepsilon)^{-1} = \sum_{i=0}^{\infty} \varepsilon^i.$$

□