

REAL ALGEBRAIC GEOMETRY II LECTURE NOTES
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CONTENTS

1.	Valued modules	1
2.	Hahn valued modules	3

Chapter I: Valued vector spaces

1. VALUED MODULES

All modules M considered are left Z -modules for a fixed ring Z with 1 (we are mainly interested in $Z = \mathbb{Z}$, i.e. in valued abelian groups).

Definition 1.1. Let Γ be a totally ordered set and ∞ an element greater than each element of Γ (Notation: $\infty > \Gamma$). A surjective map

$$v: M \longrightarrow \Gamma \cup \{\infty\}$$

is a **valuation** on M (and (M, v) is a **valued module**) if $\forall x, y \in M$ and $\forall r \in Z$:

$$(i) \quad v(x) = \infty \Leftrightarrow x = 0,$$

$$(ii) \quad v(rx) = v(x), \text{ if } r \neq 0 \text{ (value preserving scalar multiplication),}$$

$$(iii) \quad v(x - y) \geq \min\{v(x), v(y)\} \text{ (ultrametric } \Delta\text{-inequality).}$$

Remark 1.2. $(i) + (ii) \Rightarrow M$ is torsion-free.

Remark 1.3. Consequences of the ultrametric Δ -inequality:

$$(i) \quad v(x) \neq v(y) \Rightarrow v(x + y) = \min\{v(x), v(y)\},$$

$$(ii) \quad v(x + y) > v(x) \Rightarrow v(x) = v(y).$$

Definition 1.4. $v(M) := \Gamma = \{v(x) : 0 \neq x \in M\}$ is called the **value set** of M .

Definition 1.5.

- (i) Let $(M_1, v_1), (M_2, v_2)$ be valued Z -modules with value sets Γ_1 and Γ_2 respectively. Let

$$h: M_1 \longrightarrow M_2$$

be an isomorphism of Z -modules. We say that h **preserves the valuation** if there is an isomorphism of ordered sets

$$\varphi: \Gamma_1 \longrightarrow \Gamma_2$$

such that $\forall x \in M_1: \varphi(v_1(x)) = v_2(h(x))$.

- (ii) Two valuations v_1 and v_2 on M are **equivalent** if the identity map on M preserves the valuation.

Definition 1.6.

- (1) An **ordered system of Z -modules** is a pair

$$[\Gamma, \{B(\gamma) : \gamma \in \Gamma\}],$$

where $\{B(\gamma) : \gamma \in \Gamma\}$ is a family of Z -modules indexed by a totally ordered set Γ .

- (2) Two ordered systems

$$S_i = [\Gamma_i, \{B_i(\gamma) : \gamma \in \Gamma_i\}] \quad i = 1, 2$$

are **isomorphic** (we write $S_1 \cong S_2$) if and only if there is an isomorphism

$$\varphi: \Gamma_1 \longrightarrow \Gamma_2$$

of totally ordered sets, and $\forall \gamma \in \Gamma_1$ an isomorphism of Z -modules

$$\varphi_\gamma: B_1(\gamma) \longrightarrow B_2(\varphi(\gamma)).$$

- (3) Let (M, v) be a valued Z -module, $\Gamma := v(M)$. For $\gamma \in \Gamma$ set

$$M^\gamma := \{x \in M : v(x) \geq \gamma\}$$

$$M_\gamma := \{x \in M : v(x) > \gamma\}.$$

Then $M_\gamma \subsetneq M^\gamma \subsetneq M$. Set

$$B(M, \gamma) := M^\gamma / M_\gamma.$$

$B(M, \gamma)$ is called **the (homogeneous) component corresponding to γ** . **The skeleton** (*das Skelett*) of the valued module (M, v) is the ordered system

$$S(M) := [v(M), \{B(M, \gamma) : \gamma \in v(M)\}].$$

We write $B(\gamma)$ for $B(M, \gamma)$ if the context is clear.

(4) For every $\gamma \in \Gamma$, the **coefficient map** (*Koeffizient Abbildung*)

$$\begin{aligned} \pi^M(\gamma, -): M^\gamma &\longrightarrow B(\gamma) \\ x &\mapsto x + M_\gamma \end{aligned}$$

is the canonical projection.

We write $\pi(\gamma, -)$ instead of $\pi^M(\gamma, -)$ if the context is clear.

Lemma 1.7. *The skeleton is an isomorphism invariant, i.e.*

$$\begin{aligned} \text{if } (M_1, v_1) &\cong (M_2, v_2), \\ \text{then } S(M_1) &\cong S(M_2). \end{aligned}$$

Proof. Let $h: M_1 \rightarrow M_2$ be an isomorphism which preserves the valuation. Then

$$\tilde{h}: v(M_1) \longrightarrow v(M_2)$$

defined by

$$\tilde{h}(v_1(x)) := v_2(h(x))$$

is a well-defined map and an isomorphism of totally ordered sets.

For each $\gamma \in v(M_1)$, the map

$$h_\gamma: B_1(\gamma) \longrightarrow B_2(\tilde{h}(\gamma))$$

defined by

$$\pi^{M_1}(\gamma, x) \mapsto \pi^{M_2}(\tilde{h}(\gamma), h(x))$$

is well-defined and an isomorphism of modules. □

2. HAHN VALUED MODULES

A system $[\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$ of torsion-free modules can be realized as the skeleton of a valued module through the following canonical construction:

Consider $\prod_{\gamma \in \Gamma} B(\gamma)$ the product module. For $s \in \prod_{\gamma \in \Gamma} B(\gamma)$ define

$$\text{support}(s) = \{\gamma \in \Gamma : s(\gamma) \neq 0\}.$$