REAL ALGEBRAIC GEOMETRY LECTURE NOTES (15: 08/06/15 - CORRECTED ON 06/06/19)

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Contents

1. The Main Theorem

In the previous lecture we introduced the "Main Theorem" of this chapter.

Theorem 1.1. *Let $k \subseteq \mathbb{R}$ be a subfield, G a totally ordered abelian group and $\mathbb{K} := k((G))$. Then $\mathbb{K}$ is a real closed field if and only if*

(i) *G is divisible,*

(ii) *k is a real closed field.*

Last time we already proved the implication "$\Rightarrow$". For the converse we need some notions and preliminary results.

2. The divisible hull

Proposition 2.1.

(i) *Let $(G, +)$ be a torsion free abelian group. Then there exists a unique (up to isomorphism of groups) minimal divisible group $(\tilde{G}, +)$ that contains $(G, +)$.*
*$(\tilde{G}, +)$ is called the **divisible hull** of G.*

(ii) *If $H \leqslant G$, then $\tilde{H} \leqslant \tilde{G}$.*

(iii) *If G is a totally ordered abelian group (particularly torsion free), then the order on G extends uniquely to an order on $\tilde{G}$. Therefore the ordered divisible hull $(\tilde{G}, <)$ of $(G, <)$ is unique up to an order preserving isomorphism.*

Proof. (i) Consider the set $\{(x, n) : x \in G, n \in \mathbb{N}\}$ under the equivalence relation

$$(x, n) \sim (y, m) :\Leftrightarrow mx = ny,$$

i.e. set

$$\tilde{G} := \{(x,n) : x \in G, n \in \mathbb{N}\} \,/\, \sim .$$

Define an addition on $\tilde{G}$ by $(x,n)\tilde{+}(y,m) := (mx+ny, mn)$.

Verify that (ÜA)

- $\tilde{+}$ is well-defined and $(\tilde{G}, \tilde{+})$ is a torsion free abelian group.
- the map $g \mapsto (g,1)$ defines an embedding of G in $\tilde{G}$.
- $(\tilde{G}, \tilde{+})$ is divisible.
- if $G^* \supseteq G$ is a group extension and G^* is divisible and torsion free, then

  $$\mathbb{Q}G := \{qx : q \in \mathbb{Q}, x \in G\}$$

  is a minimal divisible subgroup of G^* containing G. Moreover, the map $(a,n) \mapsto \frac{1}{n}a$ is an isomorphism of groups $\tilde{G} \to \mathbb{Q}G$.

(*ii*) Straight forward by construction as in (*i*) (ÜA).

(*iii*) Declare $(x,n) \in \tilde{G}$ to be positive if and only if $x \in G$ is positive. Verify that the map $G \to \tilde{G}$, $a \mapsto (a,1)$ is order preserving.

□

Remark 2.2. *G is divisible if and only if $G = \tilde{G}$.*

Proposition 2.3. *(Generalized ultrametric inequality)*

(*i*) $v(a) \neq v(b) \Rightarrow v(a+b) = \min\{v(a), v(b)\}$.

(*ii*) $v(\sum a_i) \geqslant \min\{v(a_i)\}$.

(*iii*) *If there exists a unique index $i_0 \in \{1,\ldots,n\}$ such that $v(a_{i_0}) = \min\{v(a_i) : i = 1,\ldots,n\}$, then $v(\sum a_i) = \min\{v(a_i)\}$.*

Proposition 2.4. *Let (L,v) be a valued field and $K \subseteq L$ be a subfield such that $L|K$ is algebraic. Then $v(L)$ is contained in the divisible hull of $v(K)$.*

Proof. Let $\alpha \in v(L)\backslash v(K)$ and let $l \in L$ be such that $\alpha = v(l)$. Since L is algebraic over K, l satisfies

$$\sum_{i=0}^{n} a_i l^i = 0$$

for some $a_i \in K$ with $0 \neq a_n$. Applying v on both sides yields

$$v\left(\sum_{i=0}^{n} a_i l^i\right) = \infty = v(0).$$

Thus, there must be two indices $i,j \in \{0,\ldots,n\}$ with $i < j$ such that $\infty \neq v(a_j l^j) = v(a_i l^i)$. In other words

$$v(a_j) + jv(l) = v(a_i) + iv(l)$$

i.e.

$$(j-i)v(l) = v(a_i) - v(a_j) \in v(K)$$

and therefore

$$\alpha = \frac{v(a_i) - v(a_j)}{j-i} \in v(\tilde{K}).$$

□

3. Algebraically closed fields

In this section we prove the Main Theorem for algebraically closed fields. We conclude by showing that this transfers to real closed fields by applying the Theorem of Artin-Schreier (see RAG I).

Proposition 3.1. *Let (L, v) be a valued field and $K \subset L$ a subfield such that $L|K$ is algebraic. Then the residue field $\overline{L}$ is contained in an algebraic closure of the residue field $\overline{K}$.*

Proof. Let $0 \neq \overline{z} \in \overline{L}$ and $0 \neq z \in L$ be a preimage of $\overline{z}$ in L. Now L is algebraic over K, so z satisfies a polynomial equation

$$a_n z^n + \ldots + a_0 = 0 \quad (a_i \in K, a_n \neq 0).$$

Set $v(a_j) = \min\{v(a_i) : i = 0, \ldots, n\}$ and $b_i := \frac{a_i}{a_j}$ for $i = 0, \ldots, n$. Then $b_j = 1$ and $v(b_i) \geqslant 0$ for $i = 0, \ldots, n$. Therefore

$$0 \neq b_n X^n + \ldots + b_0 \in K_v[X]$$

and

$$b_n z^n + \ldots + b_0 = 0,$$

where K_v denotes the valuation ring of K. Thus $\overline{z}$ is a root of the non-zero polynomial $0 \neq \sum_{i=0}^n \overline{b_i} X^i \in \overline{K}[X]$, i.e. $\overline{z}$ is algebraic over $\overline{K}$. □

Theorem 3.2. *(algebraically closed fields of generalized power series, Mac Lane, 1939)*
Set $\mathbb{K} := k((G))$ for some field k and some ordered abelian group G. Then $\mathbb{K}$ is algebraically closed if and only if

(i) G is divisible,

(ii) k is an algebraically closed field.

Proof. "⇒" is analogue to the proof for the real closed field case seen last lecture (ÜA). Let us prove "⇐". So we want to show that $\mathbb{K}$ is algebraically closed.

Claim: Every algebraic extension L of $\mathbb{K}$ is immediate.

(Since $\mathbb{K}$ is maximally valued, as was shown in lectures 6 – 8, $\mathbb{K}$ will then admit no proper algebraic extensions at all, i.e. is algebraically closed)

Proof of the claim: Since $L|\mathbb{K}$ is algebraic we know by Proposition 2.4 that

$$v(L) \subseteq v(\mathbb{K}) = \tilde{G} = G.$$

On the other hand, since (L, v) is a valued extension of $(\mathbb{K}, v)$, we have $v(L) \supseteq v(\mathbb{K}) = G$, so we get $v(L) = v(\mathbb{K})$.
Similarly we show that $\overline{L} = \overline{\mathbb{K}} = k$. By Proposition 3.1 $\overline{L}$ is contained in the algebraic closure of k, but $\overline{k} = k$. So $\overline{L} \subseteq \overline{k} = k$. On the other hand, since (L, v) is a valued field extension of $(\mathbb{K}, v)$, we have $\overline{L} \supseteq \overline{\mathbb{K}} = k$, so again $\overline{L} = k$. Hence the valued field extension $(L, v)|(\mathbb{K}, v)$ is immediate. □

Remark 3.3. *What is meant in the claim is the following:* $(\mathbb{K}, v)$ *is a valued field and* $L|\mathbb{K}$ *a field extension, extending the valuation* v *on* $\mathbb{K}$ *to a valuation* v *on* L*. After that we mean* (L, v) *is an immediate extension of* $(\mathbb{K}, v)$*.*

4. Finishing the proof of the Main Theorem

Proposition 4.1. *Let* k *be a field,* G *an ordered abelian group and* $i = \sqrt{-1}$*. Then* $k((G))(i) \cong k(i)((G))$*.*

Proof. ÜA. □

Theorem 4.2. *(real closed fields of power series)*
$k((G))$ *is a real closed field if and only if* k *is a real closed field and* G *is divisible.*

Proof. It remains to prove "$\Leftarrow$". Since k is real closed, $k(i)$ is algebraically closed (Artin-Schreier). So $k(i)((G))$ is algebraically closed by Mac Lane. But then $k((G))(i)$ is also algebraically closed. By Artin-Schreier $k((G))$ is a real closed field. □

Example 4.3. Define $\tilde{\mathbb{Q}}^{\mathrm{rc}}$:= the field of all real algebraic numbers. Then $\mathbb{K} = \tilde{\mathbb{Q}}^{\mathrm{rc}}((\mathbb{Q}))$ is a real closed field. Note that $\mathbb{K}$ is not countable.

Question: Are there countable non-Archimedean real closed fields?