REAL ALGEBRAIC GEOMETRY LECTURE NOTES (09: 07/05/15 - CORRECTED ON 16/05/19)

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Contents

1. Ordered abelian groups

Definition 1.1. $(G, +, 0, <)$ is a (totally) **ordered abelian group** if $(G, +, 0)$ is an abelian group and $<$ a total order on G, such that for all $a, b, c \in G$

$$a \leqslant b \;\Rightarrow\; a + c \leqslant b + c \quad (*).$$

Definition 1.2. A subgroup C of an ordered abelian group G is **convex** if $\forall\, c_1, c_2 \in C$ and $\forall\, x \in G$

$$c_1 < x < c_2 \;\Rightarrow\; x \in C.$$

Note that because of $(*)$ this is equivalent to requiring $\forall\, c \in C$ and $\forall\, x \in G$

$$0 < x < c \Rightarrow x \in C.$$

Example 1.3. $C = \{0\}$ and $C = G$ are convex subgroups.

Lemma 1.4. *Let G be an ordered abelian group and C a convex subgroup of G. Then*

(i) *G/C is an ordered abelian group by defining $g_1+C \leqslant g_2+C$ if $g_1 \leqslant g_2$.*

(ii) *There is a bijective correspondence between convex subgroups $C \subseteq C' \subseteq G$ and convex subgroups of G/C.*

(iii) *In particular, if D and C are convex subgroups of G such that $D \subset C$ and there are no further subgroups between D and C, then C/D has no non-trivial convex subgroups.*

(iv) *If an ordered abelian group has only the trivial convex subgroups, then it is an Archimedean group.*

Definition 1.5. Let G be an ordered abelian group, $x \in G$, $x \neq 0$.
We define:

$$C_x := \bigcap \{C : C \text{ is a convex subgroup of } G \text{ and } x \in C\}.$$

$$D_x := \bigcup \{D : D \text{ is a convex subgroup of } G \text{ and } x \notin D\}.$$

A convex subgroup C of G is said to be **principal** if there is some $x \in G$ such that $C = C_x$.

Lemma 1.6.

(*i*) *C_x and D_x are convex subgroups of G.*

(*ii*) *$D_x \subsetneq C_x$.*

(*iii*) *D_x is the largest proper convex subgroup of C_x, i.e. if C is a convex subgroup such that*

$$D_x \subseteq C \subseteq C_x$$

then $C = D_x$ or $C = C_x$.

(*iv*) *It follows that the ordered abelian group C_x/D_x has no non-trivial proper convex subgroup.*

2. Archimedean groups

Definition 2.1. Let $(G, +, 0, <)$ be an ordered abelian group. We say that G is **Archimedean** if for all non-zero $x, y \in G$:

$$\exists\, n \in \mathbb{N} : \quad n|x| > |y| \text{ and } n|y| > |x|,$$

where for every $g \in G$, $|g| := \max\{g, -g\}$.

Proposition 2.2. *(Hölder) Every Archimedean group is isomorphic to a subgroup of* $(\mathbb{R}, +, 0, <)$.

Proposition 2.3. *G is Archimedean if and only if G has no non-trivial proper convex subgroup.*

Therefore if G is an ordered group and $x \in G$ with $x \neq 0$, the quotient C_x/D_x is Archimedean (by 2.3) and can be embedded in $(\mathbb{R}, +, 0, <)$ (by 2.2).

Definition 2.4. Let G be an ordered group, $x \in G$, $x \neq 0$. We say that

$$B_x := C_x/D_x$$

is **the Archimedean component** associated to x.

3. Archimedean equivalence

Definition 3.1. An abelian group G is **divisible** if for every $x \in G$ and for every $n \in \mathbb{N}$ there is some $y \in G$ such that $x = ny$.

Remark 3.2. Any ordered divisible abelian group G is an ordered $\mathbb{Q}$-vector space and G can be viewed as a valued $\mathbb{Q}$-vector space in a natural way.

Definition 3.3. (Archimedean equivalence) Let G be an ordered abelian group. For every $0 \neq x, y \in G$ we define

$$\begin{aligned} x \ \sim^+ \ y \quad &:\Leftrightarrow \quad \exists\, n \in \mathbb{N} \quad n|x| \geqslant |y| \ \text{ and } \ n|y| \geqslant |x|. \\ x \ <<^+ \ y \quad &:\Leftrightarrow \quad \forall\, n \in \mathbb{N} \quad n|x| < |y|. \end{aligned}$$

Proposition 3.4.

(1) $\sim^+$ *is an equivalence relation.*

(2) $\sim^+$ *is compatible with* $<<^+$*:*

$$\begin{aligned} x <<^+ y \quad \textit{and} \quad x \sim^+ z \quad &\Rightarrow \quad z <<^+ y, \\ x <<^+ y \quad \textit{and} \quad y \sim^+ z \quad &\Rightarrow \quad x <<^+ z. \end{aligned}$$

Because of the last proposition we can define a linear order $<_\Gamma$ on $\Gamma := G/\sim^+$, the set of equivalence classes $\{[x] : x \in G\}$, as follows:

$$\forall x, y \in G\backslash\{0\} : \ [y] <_\Gamma [x] \quad \Leftrightarrow \quad x <<^+ y \quad (\text{and } \infty > \Gamma)$$

(convention: $[0] = \infty$)

Proposition 3.5.

(1) Γ *is a totally ordered set under* $<_\Gamma$.

(2) *The map*

$$\begin{aligned} v\colon G \ &\longrightarrow \ \Gamma \cup \{\infty\} \\ 0 \ &\mapsto \ \infty \\ x \ &\mapsto \ [x] \quad (\textit{if } x \neq 0) \end{aligned}$$

is a valuation on G *as a* $\mathbb{Z}$*-module, called the* ***natural valuation****:*

For every $x, y \in G$*:*

- $v(x) = \infty$ *iff* $x = 0$,
- $v(nx) = v(x) \quad \forall\, n \in \mathbb{Z},\ n \neq 0$,
- $v(x + y) \geqslant \min\{v(x), v(y)\}$.

(3) *if* $x \in G$, $x \neq 0$, $v(x) = \gamma$, *then*

$$G^{\gamma} := \{a \in G : v(a) \geqslant \gamma\} = C_x.$$
$$G_{\gamma} := \{a \in G : v(a) > \gamma\} = D_x.$$

So

$$B_x = C_x/D_x = G^{\gamma}/G_{\gamma} = B(\gamma)$$

is the Archimedean component associated to γ. *By Hölder's Theorem, the homogeneous components* $B(\gamma)$ *are all (isomorphic to) subgroups of* $(\mathbb{R}, +, 0, <)$.

Example 3.6. Let $[\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$ be an ordered family of Archimedean groups. Consider $\coprod_{\gamma \in \Gamma} B(\gamma)$ endowed with the lexicographic order $<_{\text{lex}}$: for $0 \neq g \in \coprod_{\gamma \in \Gamma} B(\gamma)$ let $\gamma := \min \operatorname{support} g$. Then declare $g > 0 :\Leftrightarrow g(\gamma) > 0$.

Then $(\coprod B(\gamma), <_{\text{lex}})$ is an ordered abelian group. Moreover, the natural valuation is the $v_{\min}$ valuation. Similarly for the Hahn product.

Theorem 3.7. *(Hahn's embedding theorem for divisible ordered abelian groups) Let* G *be a divisible ordered abelian group with skeleton* $S(G) = [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$. *Then*

$$\left(\coprod B(\gamma), <_{\text{lex}}\right) \hookrightarrow (G, <) \hookrightarrow (\mathrm{H}\, B(\gamma), <_{\text{lex}}).$$