REAL ALGEBRAIC GEOMETRY LECTURE NOTES (04: 20/04/15 - CORRECTED ON 02/05/2019)

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Contents

Additional lecture on Ordinals

1. Preliminaries

Theorem 1.1. *(transfinite induction)*
If $(A, <)$ is a well-ordered set and $P(x)$ a property such that

$$\forall a \in A (\forall b < a \ P(b) \Rightarrow P(a)),$$

then $P(a)$ holds for all $a \in A$.

Proof. Consider the set

$$B := \{b \in A : P(b) \text{ is false}\}.$$

If $B \neq \emptyset$, let $b = \min B$. Then $\forall c < b \ P(c)$ is true but $P(b)$ is false, a contradiction. □

Definition 1.2. Let A be a well-ordered set. An **initial segment** of A is a set of the form $A_a := \{b \in A : b \leqslant a\}$.

Proposition 1.3. *No proper initial segment of a well-ordered set $(A, \leqslant)$ is $\cong A$.*

Proof. Assume $f : A \to A_a$ is an isomorphism of ordered sets. Prove by induction

$$\forall x \in A : f(x) \geqslant x.$$

Since $A_a \subsetneq A$ we find some $b \in A \backslash A_a$, i.e. $b > a$. Therefore

$$f(b) \geqslant b > a,$$

contradicting $f(b) \in A_a$. □

Definition 1.4. A set A is **transitive**, if $\forall a \in A \ \forall b \in a : b \in A$ (or equivalently $\forall a \in A : a \subseteq A$).

Lemma 1.5. *Let A be a transitive set. Then $\in$ is transitive on A if and only if a is transitive for all $a \in A$.*

Lemma 1.6. *A union of transitive sets is transitive.*

2. Ordinals

Definition 2.1. A set α is an **ordinal** if

(i) α is transitive,
(ii) $(\alpha, \in)$ is a well-ordered set.

Notation 2.2. $\text{Ord} = \{\text{ordinals}\}$

Remark 2.3. *$\in$ is an order on $\alpha \Rightarrow \in$ is transitive, i.e. $\forall a \in \alpha : a$ is transitive.*

Proposition 2.4. *$\in$ is a strict order on* Ord.

Proof. If $\alpha \in \beta \in \gamma$, then $\alpha \in \gamma$ by transitivity of γ. Therefore $\in$ is transitive on Ord. Now let $\alpha \in \beta$. We claim $\beta \notin \alpha$. Otherwise $\alpha \in \beta \in \alpha$ and therefore $\alpha, \beta \in \alpha, \alpha \in \beta, \beta \in \alpha$, a contradiction. □

We write $\alpha < \beta$ instead of $\alpha \in \beta$.

Example 2.5. Each $n \in \mathbb{N} = \{0, 1, \ldots\}$ is an ordinal

$$\begin{aligned} 0 &= \emptyset, \\ 1 &= \{0\}, \\ 2 &= \{0, 1\}, \\ 3 &= \{0, 1, 2\}, \\ &\vdots \\ n &= \{0, 1, \ldots, n-1\}. \end{aligned}$$

Moreover, $\mathbb{N} =: \omega$ is an ordinal.

Proposition 2.6. $\forall \alpha \in \text{Ord} : \alpha = \{\beta \in \text{Ord} : \beta < \alpha\}$.

Proof. Let $\beta \in \alpha$. Then β is transitive. Thus $\beta \subseteq \alpha$ and $(\beta, \in) = (\alpha, \in)_\beta$. □

Lemma 2.7. *Let $\alpha, \beta \in$ Ord such that $\alpha \subsetneq \beta$. Then $\min(\beta \backslash \alpha)$ exists and is $= \alpha$, so $\alpha \in \beta$.*

Proof. Since $\beta \backslash \alpha \neq \emptyset$, $\gamma := \min(\beta \backslash \alpha)$ exists. To show: $\gamma = \alpha$.
First let $\delta \in \gamma$, i.e. $\delta < \gamma$. Then $\delta \notin \beta \backslash \alpha$. Since $\delta \in \gamma \in \beta$, we have $\delta \in \beta$. Hence $\delta \in \alpha$.
Now let $\delta \in \alpha$. If $\delta > \gamma$, then $\alpha > \gamma$, i.e. $\gamma \in \alpha$, a contradiction. Therefore $\delta < \gamma$, i.e. $\delta \in \gamma$. □

Lemma 2.8. $\alpha \leqslant \beta \Leftrightarrow \alpha \subseteq \beta$.

Proof.

$\Rightarrow$ Clear if $\alpha = \beta$. Otherwise $\alpha < \beta$, i.e. $\alpha \in \beta$ and therefore $\alpha \subseteq \beta$ by transitivity.

$\Leftarrow$ $\alpha \subsetneq \beta \Rightarrow \alpha \in \beta \Rightarrow \alpha < \beta$.

□

Proposition 2.9. *$<$ (which is $\in$) is a total order on* Ord.

Proof. Assume $\alpha \nleq \beta$. Then $\alpha \nsubseteq \beta$. Hence $\beta \in \alpha$, i.e. $\beta < \alpha$. □

Proposition 2.10. *If $\alpha \neq \beta$, then $\alpha \ncong \beta$.*

Proof. Without loss of generality $\alpha < \beta$, so α is an initial segment of β. □

Proposition 2.11. $(\mathrm{Ord}, <)$ *is well-ordered.*

Proof. Assume $\alpha_0 > \alpha_1 > \alpha_2 > \dots$ then $(\alpha_0, <)$ is not well-ordered, a contradiction. □

Proposition 2.12.

(*i*) *If $\alpha \in \mathrm{Ord}$, then $\alpha \cup \{\alpha\} \in \mathrm{Ord}$.*
*($\alpha + 1 := \alpha \cup \{\alpha\}$ is called the **successor** of α.)*

(*ii*) *If A is a set of ordinals, then $\bigcup A \in \mathrm{Ord}$.*
*($\sup A := \bigcup A$ is the **supremum** of A.)*

Remark 2.13.

(*i*) $n + 1 = \{0, \dots, n\} = \{0, \dots, n-1\} \cup \{n\}$.

(*ii*) $\sup A$ is not always a max, e.g. $A = \{2n : n \in \omega\}$. Then $\sup A = \omega$, but A has no max.

(*iii*) If $\alpha \in \mathrm{Ord}$, then $\sup \alpha = \alpha$.

Definition 2.14. An ordinal, which is not a succesor, is called a **limit ordinal**.

Proposition 2.15. *If $\alpha \in \mathrm{Ord}$ and $P(x)$ is a property such that*

(1) *$P(0)$ is true,*

(2) $\forall \beta \in \alpha (P(\beta) \Rightarrow P(\beta + 1))$,

(3) *if $\beta \in \alpha$ is a limit ordinal, then $\forall \gamma < \beta\ P(\gamma) \Rightarrow P(\beta)$,*

Then $P(\beta)$ holds for all $\beta \in \alpha$.

Theorem 2.16. *If* $(A,<)$ *is a well-ordered set,* $\exists!\alpha \in \mathrm{Ord}, \exists!\pi : A \to \alpha$ *an isomorphism.*

Definition 2.17. This unique ordinal α is called the **order type** of A, written $\alpha = \mathrm{ot}(A)$.

Lemma 2.18. *If* $\exists\alpha \in Ord$ *such that* $A \hookrightarrow \alpha$*, then the theorem holds.*

Proof. Let $\alpha = \min\{\beta \in \mathrm{Ord} : A \hookrightarrow \beta\}$.

(1) $\pi(0) = \min A$.
(2) If $\pi(\beta)$ has been defined, either $\beta + 1 = \alpha$ (and we are done) or $\beta + 1 < \alpha$ and $A_{\pi(\beta)} \subsetneq A$. Set $\pi(\beta+1) = \min(A \backslash A_{\pi(\beta)})$.
(3) If β is a limit ordinal and if $\pi(\gamma)$ has already been defined for all $\gamma < \beta$, we distinguish two cases:
    If $\beta = \alpha$ we are done.
    If $\beta < \alpha$, set $B = \{\pi(\gamma) : \gamma < \beta\}$ and set $\pi(\beta) = \min(A \backslash B)$.

□

3. Arithmetic of ordinals

Definition 3.1. We define the **ordinal sum** $\alpha + \beta$ by induction on β :

(i) $\alpha + 0 = \alpha$,

(ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$,

(iii) if β is a limit ordinal, then $\alpha + \beta = \sup\limits_{\gamma<\beta}(\alpha + \gamma)$.

Proposition 3.2.

(i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

(ii) *If* $\beta < \gamma$*, then* $\alpha + \beta < \alpha + \gamma$.

Proof. We prove (i) by induction on γ.

- $\alpha + (\beta + 0) = \alpha + \beta = (\alpha + \beta) + 0$
-
$$\begin{aligned} \alpha + (\beta + (\gamma + 1)) &= \alpha + ((\beta + \gamma) + 1) \\ &= (\alpha + (\beta + \gamma)) + 1 \\ &= ((\alpha + \beta) + \gamma) + 1 \\ &= (\alpha + \beta) + (\gamma + 1). \end{aligned}$$
-
$$\begin{aligned} \alpha + (\beta + \gamma) &= \alpha + \sup_{\delta<\gamma}(\beta + \delta) \\ &= \sup_{\delta<\gamma}(\alpha + (\beta + \delta)) \\ &= \sup_{\delta<\gamma}((\alpha + \beta) + \delta) \\ &= (\alpha + \beta) + \gamma. \end{aligned}$$

□

Remark 3.3. *$+$ is not commutative, e.g.* $1+\omega \neq \omega+1$.

Definition 3.4. We define the **ordinal product** $\alpha \cdot \beta$ by induction on β :

(*i*) $\alpha \cdot 0 = 0$,

(*ii*) $\alpha \cdot (\beta+1) = (\alpha \cdot \beta) + \alpha$,

(*iii*) if β is a limit ordinal, then $\alpha \cdot \beta = \sup\limits_{\gamma<\beta}(\alpha \cdot \gamma)$.

Definition 3.5. We define the **ordinal exponentiation** α^β by induction on β :

(*i*) $\alpha^0 = 1$,

(*ii*) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$,

(*iii*) if β is a limit ordinal, then $\alpha^\beta = \sup\limits_{\gamma<\beta} \alpha^\gamma$.

Proposition 3.6. *Let F be the set of functions $\beta \to \alpha$ with finite support. Define*

$$f < g :\Leftrightarrow f(\gamma) < g(\gamma),$$

where $\gamma = \max\{\delta : f(\delta) \neq g(\delta)\}$. *Then* $\mathrm{ot}((F,<)) = \alpha^\beta$.

Proposition 3.7.

(*i*) $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$,

(*ii*) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$,

(*iii*) $\left(\alpha^\beta\right)^\gamma = \alpha^{\beta\cdot\gamma}$.

Remark 3.8.

(*i*) $(\omega+1)\cdot 2 \neq \omega \cdot 2 + 1 \cdot 2$,

(*ii*) $(\omega \cdot 2)^2 \neq \omega^2 \cdot 2^2$.