Universität Konstanz

Fachbereich
Mathematik und Statistik

Prof. Dr. Salma Kuhlmann
Katharina Dupont
Gabriel Lehéricy

Real Algebraic Geometry II
Exercise Sheet 12

Definition:

(a) Let A be a group and B a subgroup of A. B' is called *group complement* of B in A, if $A = B \oplus B'$.

(b) Let $G_1, \ldots, G_n$ be ordered groups. $G_1 \coprod \ldots \coprod G_n := (G_1 \oplus \ldots \oplus G_n, \leq_{\text{lex}})$ the direct product of $G_1, \ldots, G_n$ equipped with the lexicographic order.

Exercise 1 (Additive Lexicographic Decomposition)
Let $(K, \leq)$ be an ordered field. Let v be the natural valuation on K, $\overline{K}$ its residue field, G its value group $\mathcal{O}$ its valuation ring and $\mathcal{M}$ the maximal ideal of $\mathcal{O}$.

(a) Show that there exists a group complement A of $\mathcal{O}$ in K and a group complement A' of $\mathcal{M}$ in $\mathcal{O}$ such that $(K, +, 0, <) = A \coprod A' \coprod \mathcal{M}$.

(b) Show that A and A' in (a) are unique up to order isomorphism.

(c) Show that A' in (a) is order isomorphic to the archimedean group $(\overline{K}, +, 0, <)$.

(d) Show that the value set of A as in (a) is $G^{<0}$.

(e) Show that the value set of $\mathcal{M}$ is $G^{>0}$.

(f) Show that the non-zero components of A as in (a) are isomorphic to $(\overline{K}, +, 0, <)$.

(g) Show that the non-zero components of $\mathcal{M}$ are isomorphic to $(\overline{K}, +, 0, <)$.

Exercise 2
Show that the rank of an ordered field is an order isomorphism invariant, i.e. order isomorphic ordered fields have the same rank.

Exercise 3

(a) Show that $\mathbb{R}(e^x)$ is a Hardy field of rank 1.

(b) Let H be a Hardy field containing $\mathbb{R}(x)$. Assume H is closed under compositions and compositional inverses for positive infinite germs, i.e. if $f, g \in H$ then $f \circ g \in H$ and $f^{-1} \in H$.

Show that if the principal rank of H has a least element, then the rank of H is 1.

The exercise will be collected **Thursday, 09/07/2015** until 10.00 at box 13 near F 441.
