Universität Konstanz

Fachbereich
Mathematik und Statistik

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Real Algebraic Geometry II
Exercise Sheet 11

Exercise 1
Let $(K,\leq)$ be an ordered field. Let $\mathcal{O}$ be a valuation ring on K and $\mathcal{M}$ the maximal ideal of $\mathcal{O}$. Let $\mathcal{T}_{\leq}$ be the topology induced by $\leq$ given by the basis

$$\begin{aligned}\mathcal{B}_{\leq} &:= \{(a,b) \mid a,b \in K, a<b,\} \\ &:= \big\{\{x \in K \mid a<x<b\} \mid a,b \in K, a<b,\big\}\end{aligned}$$

and Let $\mathcal{T}_{\mathcal{O}}$ be the topology induced by $\mathcal{O}$ given by the basis

$$\mathcal{B}_{\mathcal{O}} := \{x \cdot \mathcal{M} + y \mid x,y \in K, x \neq 0\}.$$

(a) Assume $\mathcal{T}_{\leq} = \mathcal{T}_{\mathcal{O}}$. Show that there exists a non-trivial coarsening $\widetilde{\mathcal{O}}$ of $\mathcal{O}$, such that $\widetilde{\mathcal{O}}$ is convex in $(K,\leq)$.

(b) Assume $\mathcal{O}$ is convex in $(K,\leq)$. Show that $\mathcal{T}_{\leq} = \mathcal{T}_{\mathcal{O}}$.

Hint: Find $a,b,c,d \in K$ with $a<b$ and $c<d$, such that $(a,b) \subseteq \mathcal{M} \subseteq (c,d)$. Now find $a,b \in K$ with $a<b$, such that $(a,b) \subseteq x \cdot \mathcal{M} + y$ for $x,y \in K$ with $x>0$ and $u,v \in K$ with $u \neq 0$, such that $u \cdot \mathcal{M} + v \subseteq (c,d)$ for $c,d \in K$ with $c<d$. Conclude $\mathcal{T}_{\leq} = \mathcal{T}_{\mathcal{O}}$.

Exercise 2
Let K be a field. Let v and v' be two valuations on K with valuation rings $\mathcal{O}_v$ and $\mathcal{O}_{v'}$, maximal ideals $\mathcal{M}_v$ and $\mathcal{M}_{v'}$ and residue fields K_v and $K_{v'}$.

(a) Show that the following are equivalent

  (i) $\mathcal{O}_{v'}$ is a coarsening of $\mathcal{O}_v$.

  (ii) $\mathcal{M}_{v'} \subseteq \mathcal{M}_v$.

  (iii) For all $a,b \in K$ if $v(a) \leq v(b)$, then $v'(a) \leq v'(b)$.

(b) Assume that $\mathcal{O}_{v'}$ is a coarsening of $\mathcal{O}_v$. Let $\varphi : \mathcal{O}_{v'} \to K_{v'}$ be the residue map of v'. Let $\overline{\mathcal{O}_v} = \varphi(\mathcal{O}_v)$ be the image of $\mathcal{O}_v$ under φ. Show that $K_{v'}/K_v$ is a field extension and $\overline{\mathcal{O}_v}$ is a valuation on $K_{v'}$.

Exercise 3

(a) Let $K := \mathbb{R}((\mathbb{Q} \times \mathbb{R}))$. Consider the lexicographic order on $\mathbb{Q} \times \mathbb{R}$. Let $C = \{(0, z) \mid z \in \mathbb{R}\}$.

Compute the convex valuation associated to C, its value group and its residue field. Conclude that rank$\big(\mathbb{R}((\mathbb{Q} \times \mathbb{R}))\big) = 2$.

(b) Let $K = \mathbb{R}(t)$. Show that for any ordering on K the rank of K is a singleton with $\mathcal{R} = \{K\}$.

The exercise will be collected **Thursday, 02/07/2015** until 10.00 at box 13 near F 441.
