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VALUED FIELDS – EXERCISE 8

To be submitted on Wednesday 08.12.2010 by 14:00 in the mailbox.

Definition.

(1) An abelian group $(G,<,+)$ is called an *ordered abelian group* if $<$ is a linear order which satisfies $(x<y)\Rightarrow(x+z<y+z)$ for every $x,y,z\in G$.
(2) A field $(R,<,+,\cdot)$ is called an *ordered field* if $(R,+,<)$ if $(R,<,+)$ is an ordered abelian group and $0\leqslant x,y\Rightarrow 0\leqslant xy$.
(3) A field is called *real* if there exists a linear order $<$ such that $(R,<)$ is an ordered field. $<$ is called an ordering of R.
(4) An ordered field is called Archimedean if for every $x\in R$ there exists some $n\in\mathbb{N}$ such that $x\leqslant n$.

Question 1.

In class you showed the following theorem:

- Let $\nu : k\to\Gamma\cup\{\infty\}$ be a valuation on k, $\Gamma'\supseteq\Gamma$, $\gamma\in\Gamma'$ such that $\forall n\in\mathbb{Z}\,(n\gamma\in\Gamma\Rightarrow n=0)$, then there is a unique extension w of ν to $K(X)$ s.t. $w(X)=\gamma$.

(1) Show that without the condition that $\forall n\in\mathbb{Z}\,(n\gamma\in\Gamma\Rightarrow n=0)$, there can be more than one such extension w.
(2) Show that even if we add the condition that $\gamma\in\Gamma'\backslash\Gamma$, there can be more than one such extension w.
Hint: we consider $\mathbb{Q}((t))$ with valuation V. There are elements $a,b\in\mathbb{Q}((t))$ such that b is not algebraic over $\mathbb{Q}(a)$, $V(a)=2$, $V(b)=1$ (for instance, we may choose $a=t^2$). Let $\Gamma=2\mathbb{Z}$, and $\Gamma'=\mathbb{Z}$, $\gamma=1$. Show that we can extend $\nu:=V|_{\mathbb{Q}(a)}$ in two ways to w_1,w_2 so that there is some element of the form $m=\left(b^2+ca\right)/a$ (where $c\in\mathbb{Q}$) such that $w_1(m)=0$ and $w_2(m)>0$.
(3) Bonus: Show that in fact there are $2^{\aleph_0}$ non-equivalent extensions of $\nu:=V|_{\mathbb{Q}(t^2)}$ to $\mathbb{Q}\left(t^2\right)(X)$ such that $\nu(X)=1$.
Hint: let B be a subset of $\mathbb{Q}[[t]]$ of size $2^{\aleph_0}$ such that $\nu(a)=1$ for every $a\in B$, and even $a=t+\sum_{i=2}^{\infty}a_it^i$, and B is an algebraically independent set over t^2. So each $a\in B$ induces a valuation on $\mathbb{Q}\left(t^2\right)(X)$. Show that these are all non-equivalent.

Question 2.

Let K be a field, and let $L_1=K(X)$, $L_2=K(X,Y)$ be the fields of rational functions over K with one and two variables respectively.

(1) Define $\varphi : L_1\to K\cup\{\infty\}$ by $\varphi(f(X)/g(X))=f(0)/g(0)$ for $f,g\in K[X]$ co-prime (where $a/0=\infty$. Note that $0/0$ does not occur). Prove that it is a place, and compute its corresponding valuation (i.e. compute the valuation ring, the valuation group, the residue field, and the valuation map).
(2) Define $\psi : L_2\to L_1\cup\{\infty\}$ by $\psi(f(Y)/g(Y))=f(0)/g(0)$ for $f,g\in L_1[Y]$ co-prime as before. Show that it is also a place and compute the corresponding valuation.

(3) Define $\chi = \varphi \circ \psi : L_2 \to K \cup \{\infty\}$. Show that it is also a place, and compute its corresponding valuation.
Hint: Prove that in fact, the valuation group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ with the lexicographic order ($(a, b) < (c, d)$ iff $a < c$ or $a = c$ and $b < d$), and that the valuation map takes $X^n Y^m$ to (m, n).

Question 3.

(1) Let R be a field, and $<$ be a linear order on R such that $(R, <, +)$ is an ordered abelian group. Let $P^\times = \{x \in R \,|\, 0 < x\}$.
Show that $(R, <)$ is an ordered field iff $(P^\times, \cdot, <)$ is an ordered abelian group.
(2) Let R be an ordered field, and let $K = R((t))$. How many orderings are there on K that extend the order on R and such that the valuation ring $R[[t]]$ is convex?
(3) Compute all of these orderings explicitly, i.e. given $f(t) = \sum_{i=-n}^{\infty} a_i t^i$, write down sufficient and necessary conditions on f for f to be positive.
(4) Prove that none of these orderings is Archimedean.

Question 4.
Suppose $(R, <)$ is an ordered field which satisfies the property that $P = \{x \in R \,|\, 0 \leqslant x\}$ is contained in the set of squares
(1) Show that if $<'$ is a linear order on R so that $(R, <')$ is an ordered field then $< = <'$.
(2) Suppose in addition that R is Archimedean (for instance, R can be $\mathbb{R}$). Suppose K is a field extension of R, and that v is a valuation on K such that the residue field $\bar{K}$ is real. Show that v is trivial on R (i.e. that the valuation ring $\mathcal{O}_v$ contains R).
Hint: Use the Corollary after Baer-Krull Theorem.