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VALUED FIELDS – EXERCISE 7

To be submitted on Wednesday 15.12.2010 by 14:00 in the mailbox.

Question 1.
In Question 4, Exercise 2, we constructed $\mathbb{Q}((t))$ – the field of formal Laurant series over $\mathbb{Q}$. We showed that there is a place with $K_P = \{f \in F((t)) \mid \mathrm{supp}(f) \subseteq \mathbb{N} = \{0, 1, \ldots\}\}$. Compute the corresponding valuation (i.e. the value map, the valuation group and the residue field).

Question 2.
Let $L = \mathbb{Q}[[t]]^{\times}$ be the group of units in the valuation ring of $\mathbb{Q}((t))$. In Exercise 3 we defined the notion of transcendental degree over $\mathbb{Q}$ (just for a ring, but really for any set). Prove that $\mathrm{tr.deg}_{\mathbb{Q}}(L) = 2^{\aleph_0} > \aleph_0$.
Hints:

(1) Recall that for any field k, and $K \supseteq k$, for a set $X \subseteq K$, $\mathrm{cl}_k(X) = \{y \in K \mid y \text{ is algebraic over } k(X)\}$. Show that $|\mathrm{cl}_k(X)| \leqslant |k| + \aleph_0 + |X|$ (use the fact that for 2 infinite sets A, and B, $\sum_{n \in \mathbb{N}} |A|^n = |A| + \aleph_0$ and $|A| + |B| = |A| \cdot |B| = \max(|A|, |B|)$).
(2) Conclude that if $\mathrm{tr.deg}_{\mathbb{Q}}(L) < 2^{\aleph_0}$, then $|L| < 2^{\aleph_0}$.
(3) Look at the definition of L and derive a contradiction.

Question 3.

(1) Let R be a Dedekind Domain, and let $K = \mathrm{quot}(R)$ be its field of fractions. Let v be a valuation on K such that its valuation ring K_v contains R. Show that v is a $\mathfrak{p}$-adic valuation for some prime ideal $\mathfrak{p}$ of R.
Let K be a field.
(2) Classify all valuations on $K(X)$ that are trivial on K.
Hint: Let w be a valuation on $K(X)$. Either the valuation ring $K_w \supseteq K[X]$, or $w(x) < 0$, in which case $K_w \supseteq K[1/x]$. Use (1).

In class you showed the following theorem:
- Let $v : k \to \Gamma \cup \{\infty\}$ be a valuation on k. Then there is a unique extension w of v to $K(X)$ s.t. $w(X) = 0$ and $\bar{X}$ is transcendental over $\bar{K}$.

(3) Without the condition that $\bar{X}$ is transcendental over $\bar{K}$, there is more than one such valuation w. In fact, there can be infinitely many extensions.
Hint: Use (2) to find infinitely many such valuations.

Question 4.
Let V be the valuation of the field of Laurent series $\mathbb{Q}((t))$. Show that the number of non-equivalent valuations w on $\mathbb{Q}(t)(X)$ extending $V|_{\mathbb{Q}(t)}$ with $w(X) = 0$ is $2^{\aleph_0}$ (Hint: use Question 2 to find a set $B \subseteq L$ of size $2^{\aleph_0}$ such that every $a \in B$ is not algebraic over $\mathbb{Q}(t)$. For every $a \in B$, $\mathbb{Q}(t)(a)$ is isomorphic to $\mathbb{Q}(t)(X)$, and it induces a valuation on $\mathbb{Q}(t)(X)$, call it v_a. Show that all these valuations are non

equivalent: for every $a \neq b \in B$, find a polynomial $p_{a,b}(t)$ over $\mathbb{Q}$ and a number $m \in \mathbb{N}$ such that $v_a((X - p_{a,b}(t))/t^m) > 0$ while $v_b((X - p_{a,b}(t))/t^m) = 0$ or vice-versa.