Prof. Dr. Salma Kuhlmann Dr. Itay Kaplan

VALUED FIELDS – EXERCISE 6

To be submitted on Wednesday 08.12.2010 by 14:00 in the mailbox.

Definition.

(1) A valuation ν on a field K is a group homomorphism from $K^{\times}$ into an ordered abelian group $(\Gamma, <)$ such that $\nu(x+y) \geqslant \min(\nu(x), \nu(y))$.
(2) Given a valuation ν on K, $K_\nu := \{x \,|\nu(x) \geqslant 0\}$ is the valuation ring, $m_\nu = \{x \,|\nu(x) > 0\}$ is the maximal ideal, and $k_\nu = K_\nu / m_\nu$ is the residue field.

Question 1.
Suppose $K \subseteq L$ is an algebraic extension, P is a place on K and λ is a place on L such that λ extends P (i.e. $K_P = L_\lambda \cap K$).

(1) Show that $m_P = K \cap m_\lambda$ (where m_P, m_λ are the maximal ideals of K_P, L_λ resp.).
(2) Let C be the integral closure of K_P in L, and let $l = C \cap m_\lambda$. Show that l is a maximal ideal in C (Hint: Question 4, (4)).
(3) Show that $C_l \subseteq L_\lambda$ (C_l is the localization of C in l).
In the rest of the question, you will show that $C_l = L_\lambda$.
(4) Given $t \in L$, show that there is some polynomial $p(X) \in K_P[X]$ such that $p(t) = 0$ and moreover, if $p(X) = \sum_{i \leqslant n} a_i X^i$ then there is some $j < n$ such that $a_j = 1$, and $P(a_i) = 0$ for all $i < j$.
(5) Suppose that $t \in M_\lambda$, and $p(t) = 0$ is as in (5). Let $a = a_n t^{n-j} + \ldots + a_j, b = a_{j-1} + a_{j-2}/t + \ldots + a_0/t^{j-1}$ (you may assume $a_{-1} = 0$) so that $at + b = 0$. Show that $a, b \in C$.
Hint: use the theorem saying that the integral closure of a ring R in a field F is the intersection of all valuation rings containing R.
(6) Prove that $a \notin l$, and finish.

Question 2.
Suppose $K \subseteq L$ is an algebraic extension

(1) Deduce from Question 1 that if ν is a valuation on L such that $\nu|_K$ is trivial, then ν is trivial.
(2) Prove (1) directly from the definitions of a valuation.

Question 3.
Translate the following statements from places to valuations, and explain how they follow from what you have seen in the course:

(1) Let $K_2 \supseteq K_1$ be a field extension and let ν be a valuation on K_1. Then there is a valuation ν_2 on K_2 such that ν_2 extends ν_1 (i.e. $\Gamma_{\nu_2} \supseteq \Gamma_{\nu_1}$ and $\nu_2|_{K_1} = \nu_1$).
(2) Under the assumption of the clause (1), prove that there is a valuation ν_2 such that k_{ν_2} is an algebraic extension of k_{ν_1}.

Question 4.

Let R be a Dedekind domain. Let K be its quotient field.

(1) Show that all local subrings of K containing R are valuation rings.
(2) Suppose A is a local subring of K, but now it does not necessarily contains R, is (1) still true?
Hint: consider $\mathbb{F}_p(X)$.
(3) Now let k be a field, $R = k[X]$ and $K = k(X)$. Consider the valuation $v : K^\times \to \mathbb{Z}$ defined as $v(f/g) = \deg(f) - \deg(g)$. Show that v is a valuation, and compute the valuation ring, and show that it does not contain R.

Question 4.

Let K be a field, and v a valuation on it. Consider the Gauss extension of v to $L := K(X)$ defined by $v\left(\sum a_i x^i\right) = \min\{v(a_i)\}$, and $v(f/g) = v(f) - v(g)$ for $f, g \in K[X]$. In class it was shown to be a valuation.
Let l_v be the residue field of L, and k_v the residue field of K, and let $\pi_K : K_v \to k_v$ be the residue map of K to the residue field (this is the corresponding place), and let $\pi_L : L_v \to l_v$ be the residue map of L.
Prove that $\pi_L(X)$ is transcendental over k_v and that l_v is $k_v(\pi_L(X))$.