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VALUED FIELDS – EXERCISE 5

To be submitted on Wednesday 01.12.2010 by 14:00 in the mailbox.

Definition.

(1) Recall: for a ring (not necessarily a domain) R, a multiplicative set $S \subseteq R$ is a set that contains 1, and for all $x, y \in S$, $xy \in S$. For such a set we define $S^{-1}R$ as the ring whose elements are $\sim$ equivalence classes of pairs $(x, y) \in R \times S$ where $(x, y) \sim (x', y')$ iff $\exists u \in S\,(u\,(xy' - x'y) = 0)$. You may think of elements in $S^{-1}R$ as x/y where $x \in R$, $y \in S$. Addition and multiplication are defined as usual: $(x, y) + (x', y') = (xy' + x'y, yy')$, $(x, y)\,(x', y') = (xx', yy')$.

(2) Let $A \subseteq B$ be rings. An element $\alpha \in B$ is said to be integral over A if it satisfies a monic polynomial with coefficients from A: if there exists a relation of the form $\alpha^m + a_1\alpha^{m-1} + \cdots + a_m = 0$ where $a_i \in A$.

(3) The ring $B \supseteq A$ is an integral extension of A if all its elements are integral over A.

Question 1.

Suppose R is a ring, S a multiplicative subset of R.

(1) Prove that $\sim$ is an equivalence relation, that $+, \cdot$ are well defined on $S^{-1}R$ are well defined, that $S^{-1}R$ is a ring, and that the map $x \mapsto x/1$ from R to S is a ring homomorphism.

(2) Show that there is a 1-1 correspondence between $\{Q\,|Q \text{ is prime in } R, Q \cap S = \emptyset\}$ and prime ideals in $S^{-1}R$: $\mathfrak{p}$ is matched with $S^{-1}\mathfrak{p} := \{(x, y)\,|x \in \mathfrak{p}, y \in S\}$.

Question 2.

Suppose R is a Dedekind Domain (recall the definition from Exercise 1). Suppose P is a prime ideal in R $(0 \neq P \neq R)$.

(1) Show that $\bigcap_{i=1}^{\infty} P^i = 0$.
Hint: use the results from Exercise 1.

(2) Let R_P be the localization in P. Deduce that there is $m \in R_P$ such that every element in R_P can be written as um^q where u is a unit, $q \in \mathbb{N}$.
Hint: Show that $(PR_P)^2 \subsetneq PR_P$.

Question 3.

Suppose $A \subseteq B$ are integral domains, and that B is an integral extension of A.

(1) Suppose S is a multiplicative subset of A. Show that $S^{-1}B$ is an integral extension of $S^{-1}A$.

(2) Suppose $\mathfrak{q}$ is a prime ideal of B and $\mathfrak{p}$ is a prime ideal of A such that $\mathfrak{p} = \mathfrak{q} \cap A$. Show that $B/\mathfrak{q}$ is an integral extension of $A/\mathfrak{p}$.

(3) Prove that A is a field iff B is a field.
Hint: Suppose B is a field and A is not. Use Corollary V to Theorem 5 (Chevalley). Another option is: suppose $0 \neq x \in A$, then $x^{-1} \in B$, so there is a monic polynomial $f(X)$ such that $f\left(x^{-1}\right) = 0$. Manipulate this

equation to show that $x^{-1} \in A$. The second direction follows directly from the definitions.

(4) Now suppose as above that $\mathfrak{q}$ is a prime ideal of B and $\mathfrak{p}$ is a prime ideal of A such that $\mathfrak{p} = \mathfrak{q} \cap A$. Conclude that $\mathfrak{p}$ is maximal iff $\mathfrak{q}$ is.

(5) Conclude that if $\mathfrak{q}_1, \mathfrak{q}_2$ are 2 prime ideals of B such that $\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A = \mathfrak{p}$ and $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ then $\mathfrak{q}_1 = \mathfrak{q}_2$.
Hint: Suppose not. Let $S = A \backslash \mathfrak{p}$. Then $S^{-1}A = A_{\mathfrak{p}}$, and $S^{-1}B$ is integral over $A_{\mathfrak{p}}$. Also $S^{-1}\mathfrak{q}_1 \subsetneq S^{-1}\mathfrak{q}_2$ are different prime ideals of $S^{-1}B$ (why?), such that the intersection with $A_{\mathfrak{p}}$ is $\mathfrak{p}A_{\mathfrak{p}}$. Use (4).

Question 4.

Prove the following:

Let $A \subseteq B$ be integral domains. Assume:

- B is integral over A.
- P is a prime ideal of B.
- $\mathfrak{p}$ is a prime ideal of A.
- $\mathfrak{p} = P \cap A$.
- $\mathfrak{q} \supseteq \mathfrak{p}$ is another prime ideal in A.

Then there exists a prime ideal Q of B such that $Q \supseteq P$ and $Q \cap A = \mathfrak{q}$.
Hint: see Corollary V to Theorem 5 (Chevalley), use Question 3.